Using Dimension to Find NS(A) & CS(A)

Linear Algebra MATH 2076



Let's "find" $\mathcal{NS}(A)$ and $\mathcal{CS}(A)$ where

$$A = \begin{bmatrix} \vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4 \ \vec{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

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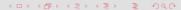
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Therefore, rank(A) = 3, and hence $CS(A) = \mathbb{R}^3$.

Since rank(A) + null(A) = 5, we now get null(A) = 5 - 3 = 2.

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Thus $\{\vec{z}_1, \vec{z}_2\}$ is a LI set of vectors in the 2-dimensional space $\mathcal{NS}(A)$, so it is a basis. We conclude that $\mathcal{NS}(A) = \mathcal{S}pan\{\vec{z}_1, \vec{z}_2\}$, a 2-plane in \mathbb{R}^5 .

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 $\mathcal{CS}(A)=\mathbb{R}^3$ (so, $A\vec{x}=\vec{b}$ has a solution for every \vec{b} in \mathbb{R}^3), and $\mathcal{NS}(A)$ is the 2-plane in \mathbb{R}^5 that is given by

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