Using Dimension to Find $\mathcal{NS}(A)$ & $\mathcal{CS}(A)$

Linear Algebra
MATH 2076

The Problem and Our Strategy

Let's "find" $\mathcal{NS}(A)$ and $\mathcal{CS}(A)$ where

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

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As A is a 3×5 matrix, we already know:

- $\mathcal{NS}(A)$ is a vector subspace of $\mathbb{R}^5$
- $\mathcal{CS}(A)$ is a vector subspace of $\mathbb{R}^3$.

Finding the Column Space $\mathcal{CS}(A)$

We already know $\mathcal{CS}(A)$ is a vector subspace of $\mathbb{R}^3$.

$$A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5\end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Finding the Column Space $\mathcal{CS}(A)$

We already know $\mathcal{CS}(A)$ is a vector subspace of $\mathbb{R}^3$.

Therefore, $\text{rank}(A) = \dim \mathcal{CS}(A) \leq 3$.

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We already know $\mathcal{CS}(A)$ is a vector subspace of $\mathbb{R}^3$.

Therefore, $\text{rank}(A) = \dim \mathcal{CS}(A) \leq 3$.

Notice that $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is clearly LI.

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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Notice that $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is clearly LI. Thus this is a maximal LI set in $\mathcal{CS}(A)$, so

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Notice that $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is clearly LI. Thus this is a maximal LI set in $\mathcal{CS}(A)$, so it is a basis.

Therefore, $\text{rank}(A) = 3$, and hence $\mathcal{CS}(A) =$

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Therefore, $\text{rank}(A) = 3$, and hence $\mathcal{CS}(A) = \mathbb{R}^3$.

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Since $\text{rank}(A) + \text{null}(A) = 5$, we now get $\text{null}(A) =$

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We already know $\mathcal{CS}(A)$ is a vector subspace of $\mathbb{R}^3$.

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Notice that $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is clearly LI. Thus this is a maximal LI set in $\mathcal{CS}(A)$, so it is a basis.

Therefore, $\text{rank}(A) = 3$, and hence $\mathcal{CS}(A) = \mathbb{R}^3$.

Since $\text{rank}(A) + \text{null}(A) = 5$, we now get $\text{null}(A) = 5 - 3 = 2$.

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Thus $\{\vec{z}_1, \vec{z}_2\}$ is a LI set of vectors in the 2-dimensional space $\mathcal{NS}(A)$, so

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

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Thus $\{\vec{z}_1, \vec{z}_2\}$ is a LI set of vectors in the 2-dimensional space $\mathcal{NS}(A)$, so it is a basis.

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

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Thus $\{\vec{z}_1, \vec{z}_2\}$ is a LI set of vectors in the 2-dimensional space $\mathcal{NS}(A)$, so it is a basis. We conclude that $\mathcal{NS}(A) = \mathcal{S}pan\{\vec{z}_1, \vec{z}_2\}$, a 2-plane in $\mathbb{R}^5$.

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

Conclusion

For the matrix

$$A = \left[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4\ \vec{a}_5\right] = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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$\mathcal{CS}(A) = \mathbb{R}^3$ (so, $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^3$), and

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For the matrix

$$A = \left[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4\ \vec{a}_5\right] = \begin{bmatrix} 1 & 3 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$\mathcal{CS}(A) = \mathbb{R}^3$ (so, $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^3$), and $\mathcal{NS}(A)$ is the 2-plane in $\mathbb{R}^5$ that is given by

$$\mathcal{NS}(A) = \mathcal{S}pan\{\vec{z}_1, \vec{z}_2\} = \mathcal{S}pan\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$