Linear Algebra MATH 2076



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The (i,j)-minor of A is the $(n-1)\times (n-1)$ matrix M_{ij} obtained by deleting both the i^{th} row and j^{th} column of A:

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\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}
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 E.g., the (2,3) minor of
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
 is
$$\begin{bmatrix} a & b \\ g & h \end{bmatrix}.$$

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$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j})$$

$$= a_{11} |M_{11}| - a_{12} |M_{12}| + \dots + (-1)^{1+n} a_{1n} |M_{1n}|.$$

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Section 3.2 Properties of Dets 17 February 2017

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 (cofactor expansion across the i^{th} row)

or by expanding down any column

Section 3.2

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(M_{ij})$$
 (cofactor expansion down the j^{th} column).

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- Multiply one row by a non-zero constant.

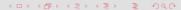
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Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$. Suppose we perform an elem row op on A to get B. Then:

• det(B) = -det(A) for a type (3) elem row op



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- det(B) = k det(A) for a type (2) elem row op
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What is the effect of these elem row ops on the determinant?

- det(B) = det(A) for a type (1) elem row op
- det(B) = k det(A) for a type (2) elem row op
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- det(B) = k det(A) for a type (2) elem row op
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Examples

Find the determinants of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 & 4 \\ 3 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

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Find the determinants of

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Answers: det(A) = 18 and det(B) = -100.



Let A and B be square matrices of the same size. Then:

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- $det(A^T) = det(A)$

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- $det(kA) = k^n det(A)$ (if A is $n \times n$)
- If A is invertible, then $det(A^{-1}) = (det(A))^{-1}$