Properties of Determinants

Linear Algebra
MATH 2076

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of certain $(n-1) \times (n-1)$ matrices called the *minors* of A.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of certain $(n-1) \times (n-1)$ matrices called the *minors* of A.

The (i,j)-*minor of* A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting both the i^{th} row and j^{th} column of A:

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of certain $(n-1) \times (n-1)$ matrices called the *minors* of A.

The (i, j)-*minor of* A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting both the i^{th} row and j^{th} column of A:

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

E.g., the $(2, 3)$ minor of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $\begin{bmatrix} a & b \\ g & h \end{bmatrix}$.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\left|M_{11}\right| - a_{12}\left|M_{12}\right| + \cdots + (-1)^{1+n} a_{1n}\left|M_{1n}\right|.\end{aligned}$$

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\big|M_{11}\big| - a_{12}\big|M_{12}\big| + \cdots + (-1)^{1+n} a_{1n}\big|M_{1n}\big|.\end{aligned}$$

This is called *cofactor expansion across the first row*.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\left|M_{11}\right| - a_{12}\left|M_{12}\right| + \cdots + (-1)^{1+n} a_{1n}\left|M_{1n}\right|. \end{aligned}$$

This is called *cofactor expansion across the first row*. In fact, we can calculate $\det(A)$ by expanding across any row

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\left|M_{11}\right| - a_{12}\left|M_{12}\right| + \cdots + (-1)^{1+n} a_{1n}\left|M_{1n}\right|.\end{aligned}$$

This is called *cofactor expansion across the first row*. In fact, we can calculate $\det(A)$ by expanding across any row

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion across the } i^{\text{th}} \text{ row)}$$

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\left|M_{11}\right| - a_{12}\left|M_{12}\right| + \cdots + (-1)^{1+n} a_{1n}\left|M_{1n}\right|.\end{aligned}$$

This is called *cofactor expansion across the first row*. In fact, we can calculate $\det(A)$ by expanding across any row

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion across the } i^{\text{th}} \text{ row)}$$

or by expanding down any column

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion down the } j^{\text{th}} \text{ column).}$$

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero.

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\ldots a_{nn}.$$

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

The following are allowable elementary row operations.

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

The following are allowable elementary row operations.

- Add a multiple of one row to another.

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

The following are allowable elementary row operations.

- Add a multiple of one row to another.
- Multiply one row by a *non-zero* constant.

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\ldots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

The following are allowable elementary row operations.

- Add a multiple of one row to another.
- Multiply one row by a *non-zero* constant.
- Interchange two rows.

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

The following are allowable elementary row operations.

- Add a multiple of one row to another.
- Multiply one row by a *non-zero* constant.
- Interchange two rows.

What is the effect of these elem row ops on the determinant?

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

What is the effect of these elem row ops on the determinant?

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

What is the effect of these elem row ops on the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$.

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

What is the effect of these elem row ops on the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$.
Suppose we perform an elem row op on A to get B. Then:

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

What is the effect of these elem row ops on the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(M_{ij})$.
Suppose we perform an elem row op on A to get B. Then:

- $\det(B) = -\det(A)$ for a type (3) elem row op

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

What is the effect of these elem row ops on the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$.
Suppose we perform an elem row op on A to get B. Then:

- $\det(B) = k \det(A)$ for a type (2) elem row op
- $\det(B) = -\det(A)$ for a type (3) elem row op

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

What is the effect of these elem row ops on the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$.
Suppose we perform an elem row op on A to get B. Then:

- $\det(B) = \det(A)$ for a type (1) elem row op
- $\det(B) = k \det(A)$ for a type (2) elem row op
- $\det(B) = -\det(A)$ for a type (3) elem row op

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

What is the effect of these elem row ops on the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$.
Suppose we perform an elem row op on A to get B. Then:

- $\det(B) = \det(A)$ for a type (1) elem row op ($\ddot\smile$)
- $\det(B) = k \det(A)$ for a type (2) elem row op
- $\det(B) = -\det(A)$ for a type (3) elem row op

Examples

Find the determinants of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 & 4 \\ 3 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Examples

Find the determinants of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 & 4 \\ 3 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Answers: $\det(A) = 18$ and $\det(B) = -100$.

Properties of Determinants

Let A and B be square matrices of the same size. Then:

Properties of Determinants

Let A and B be square matrices of the same size. Then:

- $\det(AB) = \det(A)\det(B)$

Properties of Determinants

Let A and B be square matrices of the same size. Then:

- $\det(AB) = \det(A)\det(B)$
- $\det(kA) = k^n \det(A)$ (if A is $n \times n$)

Properties of Determinants

Let A and B be square matrices of the same size. Then:

- $\det(AB) = \det(A)\det(B)$
- $\det(kA) = k^n \det(A)$ (if A is $n \times n$)
- $\det(A^T) = \det(A)$

Properties of Determinants

Let A and B be square matrices of the same size. Then:

- $\det(AB) = \det(A)\det(B)$
- $\det(kA) = k^n \det(A)$ (if A is $n \times n$)
- $\det(A^T) = \det(A)$
- If A is invertible, then $\det(A^{-1}) = \big(\det(A)\big)^{-1}$