Properties of Determinants

Linear Algebra
MATH 2076

The determinant of an $n \times n$ matrix A is given in terms of determinants of certain $(n-1) \times (n-1)$ matrices called the *minors* of A.

The (i, j)-*minor of* A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting both the i^{th} row and j^{th} column of A:

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

E.g., the $(2, 3)$ minor of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $\begin{bmatrix} a & b \\ g & h \end{bmatrix}$.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\left|M_{11}\right| - a_{12}\left|M_{12}\right| + \cdots + (-1)^{1+n} a_{1n}\left|M_{1n}\right|.\end{aligned}$$

This is called *cofactor expansion across the first row*. In fact, we can calculate $\det(A)$ by expanding across any row

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion across the } i^{\text{th}} \text{ row)}$$

or by expanding down any column

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion down the } j^{\text{th}} \text{ column).}$$

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

The following are allowable elementary row operations.

- Add a multiple of one row to another.
- Multiply one row by a *non-zero* constant.
- Interchange two rows.

How do these elem row ops change the determinant?

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

How do these elem row ops change the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$.
Suppose we perform an elem row op on A to get B. Then:

- $\det(B) = \det(A)$ for a type (1) elem row op ($\ddot\smile$)
- $\det(B) = k \det(A)$ for a type (2) elem row op
- $\det(B) = -\det(A)$ for a type (3) elem row op

Examples

Find the determinants of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 & 4 \\ 3 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Answers: $\det(A) = 18$ and $\det(B) = -100$.

Properties of Determinants

Let A and B be square matrices of the same size. Then:

- $\det(AB) = \det(A)\det(B)$
- $\det(kA) = k^n\det(A)$ (if A is $n \times n$)
- $\det(A^T) = \det(A)$
- If A is invertible, then $\det(A^{-1}) = \left(\det(A)\right)^{-1}$