Determinants—an Introduction

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

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Calculating $\det(A)$ is **not** good way to determine if A is invertible!
See "Numerical Note" on page 169 of text.

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It's convenient to write $|A| = \det(A)$. So, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2$.

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The (i,j)-*minor of* A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting both the i^{th} row and j^{th} column of A:

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

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E.g., the $(2, 3)$ minor of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $\begin{bmatrix} a & b \\ g & h \end{bmatrix}$.

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$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\left|M_{11}\right| - a_{12}\left|M_{12}\right| + \cdots + (-1)^{1+n} a_{1n}\left|M_{1n}\right|. \end{aligned}$$

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$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion across the } i^{\text{th}} \text{ row)}$$

or by expanding down any column

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion down the } j^{\text{th}} \text{ column).}$$

Example

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$