Determinants—an Introduction

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

What is a Determinant?

For each square matrix A, get associated number $\det(A)$ with properties:

- A is invertible if and only if $\det(A) \neq 0$

- $\det(A) = \pm\text{vol}(\Pi)$ where Π is image of unit cube under $\vec{x} \mapsto A\vec{x}$

Thus have function $\mathbb{R}^{n \times n} \xrightarrow{\det} \mathbb{R}$ where $A \mapsto \det(A)$.

Calculating $\det(A)$ is a **terrible** way to determine if A is invertible!
See "Numerical Note" on page 169 of text.

What is a Determinant?

The determinant function $\mathbb{R}^{n \times n} \xrightarrow{\det} \mathbb{R}$ is defined recursively.

To find the determinant of an $n \times n$ matrix, we need to know how to find the determinant of an $(n-1) \times (n-1)$ matrix.

The determinant of a 2×2 matrix is easy to calculate:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

For example,

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2.$$

It's convenient to write $|A| = \det(A)$. So, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2.$

The Determinant of a 3×3 Matrix

The determinant of a 3×3 matrix is given by

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - ge)$$

This is called *cofactor expansion across the first row*. For example,

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9 = 0.$$

The Determinant of a 4×4 Matrix

The determinant of a 4×4 matrix is given by

$$
\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix} =
$$

$$
= a \begin{vmatrix} f & g & h \\ j & k & l \\ n & p & q \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & p & q \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & q \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & p \end{vmatrix}.
$$

This is called *cofactor expansion across the first row*.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of certain $(n-1) \times (n-1)$ matrices called the *minors* of A.

The (i,j)-*minor of* A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting both the ith row and j^{th} column of A:

$$\begin{bmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \ldots & a_{nj} & \ldots & a_{nn} \end{bmatrix}$$

E.g., the $(2,3)$ minor of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $\begin{bmatrix} a & b \\ g & h \end{bmatrix}$.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j})$$
$$= a_{11} |M_{11}| - a_{12} |M_{12}| + \cdots + (-1)^{1+n} a_{1n} |M_{1n}|.$$

This is called *cofactor expansion across the first row*. In fact, we can calculate $\det(A)$ by expanding across any row

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion across the } i^{\text{th}} \text{ row)}$$

or by expanding down any column

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion down the } j^{\text{th}} \text{ column)}.$$

Example

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$