Subspaces of Euclidean Space $\mathbb{R}^n$
Finding Null Space and Column Space

Linear Algebra
MATH 2076

"Finding" Null Space and Column Space

To "find" the null space $\mathcal{NS}(A)$ and column space $\mathcal{CS}(A)$ of a matrix A:

- row reduce A to E, a REF (or RREF) for A
- columns of E containing row leaders correspond to *pivot* columns of A
- the *pivot* columns of A are LI and span $\mathcal{CS}(A)$
- write the SS for $A\vec{x} = \vec{0}$ in parametric vector form
- identify LI vectors that span $\mathcal{NS}(A)$

So, "find" means to find a *linearly independent spanning* set.

A *linearly independent spanning* set for a vector subspace is called a *basis*.

Example—Null Space and Column Space

Find bases for the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using elem row ops, we find the indicated REF and RREF for A. Thus columns 1,2,4 are pivot columns for A, so a basis for $\mathcal{CS}(A)$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ and we see that $\mathcal{CS}(A)$ is a 3-plane in $\mathbb{R}^4$.

Let's focus on $\mathcal{NS}(A)$. So, we need to "solve" $A\vec{x} = \vec{0}$. The free variables are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Example—Null Space and Column Space

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{NS}(A)$ is a vector subspace of $\mathbb{R}^5$. To "find" $\mathcal{NS}(A)$, we solve $A\vec{x} = \vec{0}$.
Free vrbls are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Thus $A\vec{x} = \vec{0}$ iff $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s + t \\ -s + 2t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

So, $\mathcal{NS}(A)$ is a 2-plane in $\mathbb{R}^5$ and the above two vectors form a basis.