Subspaces of Euclidean Space $\mathbb{R}^n$

Linear Algebra
MATH 2076

University of Cincinnati

Vector Subspaces—Basic Example

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Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a vector subspace.

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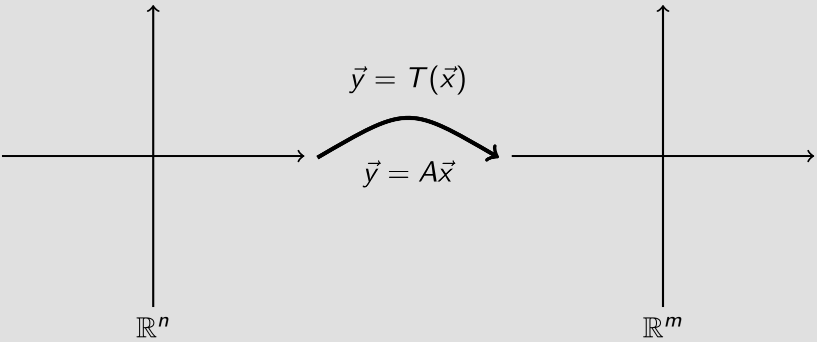

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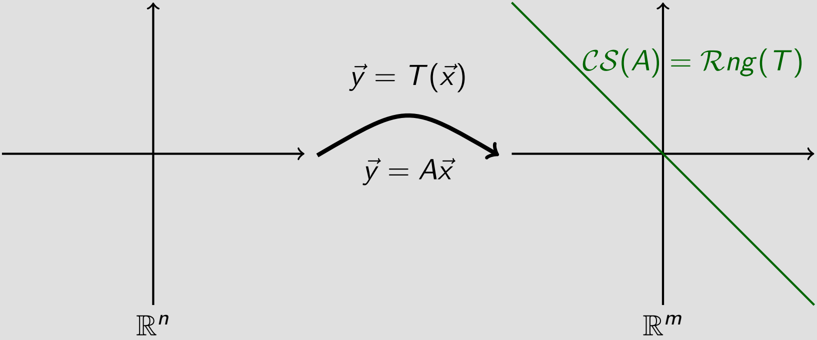

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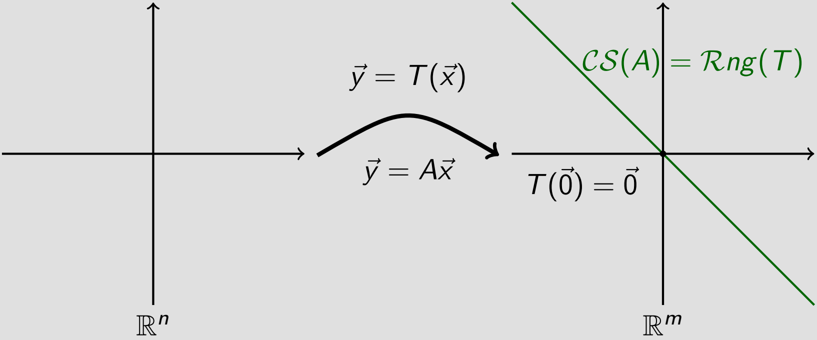

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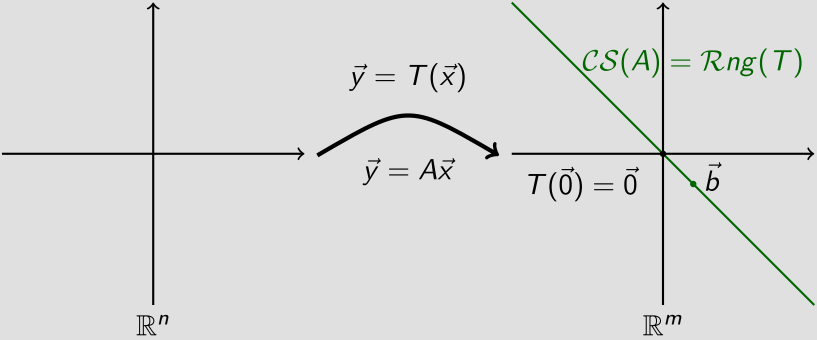

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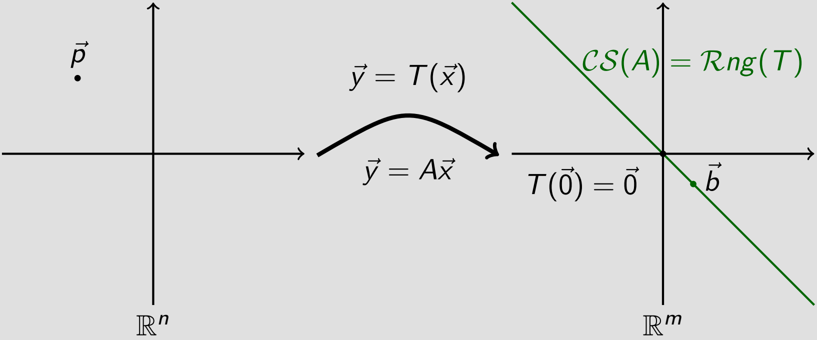

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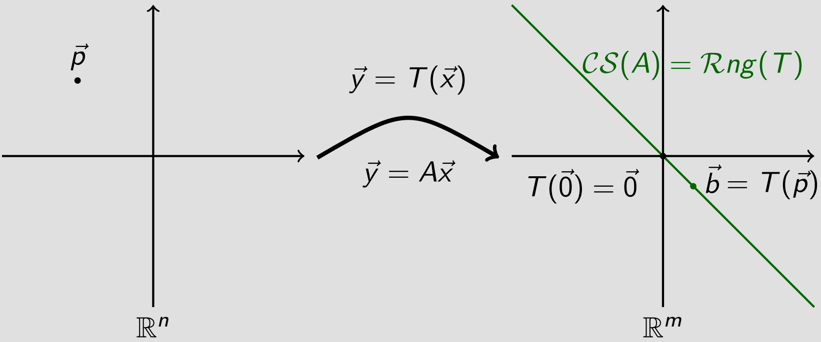

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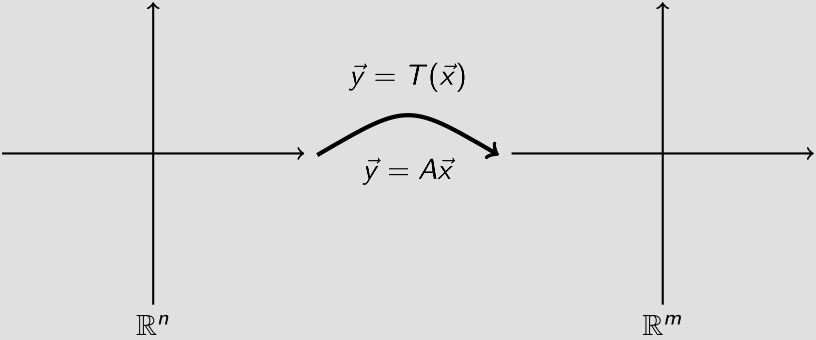

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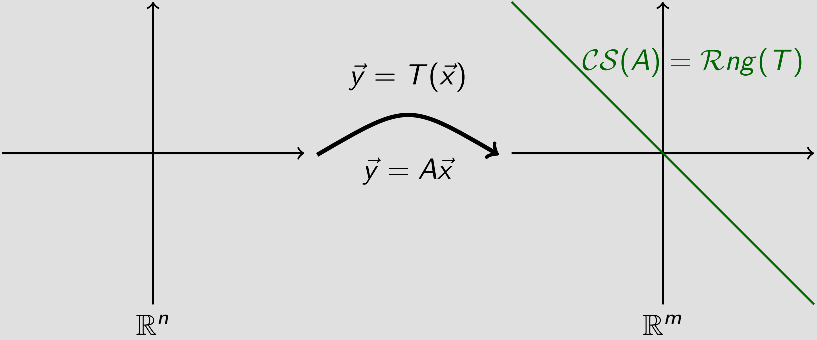

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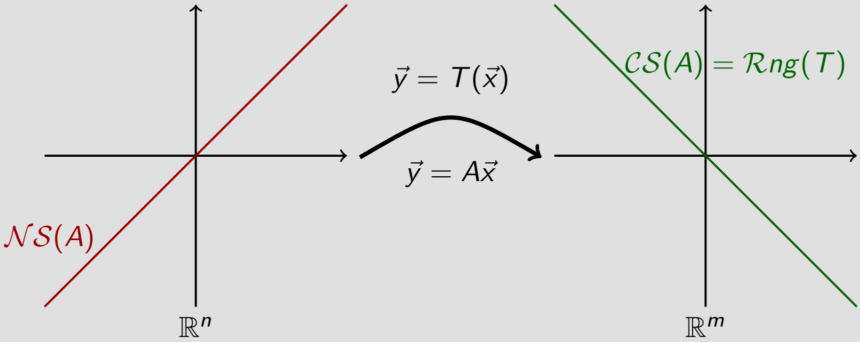

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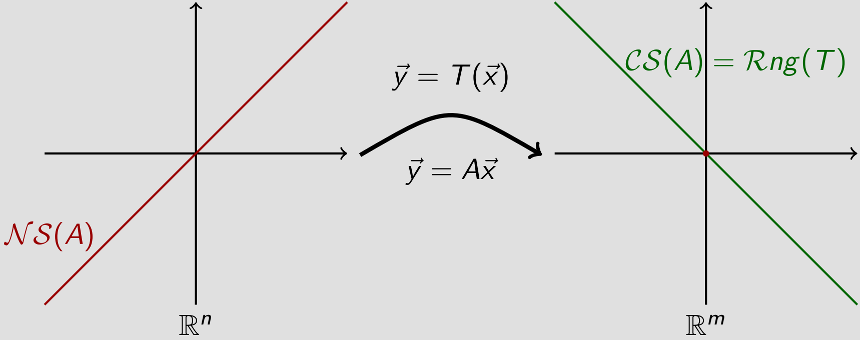

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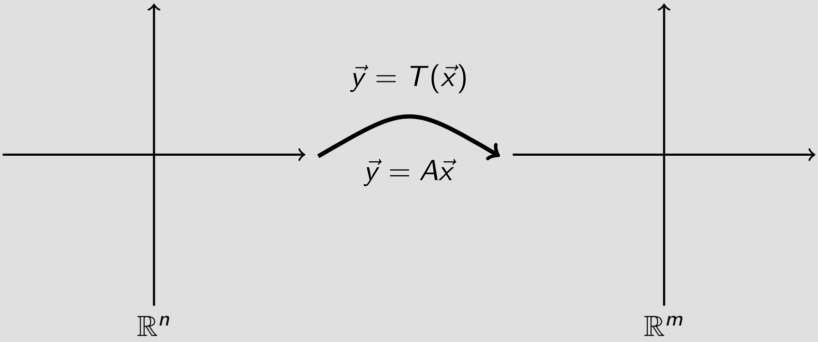

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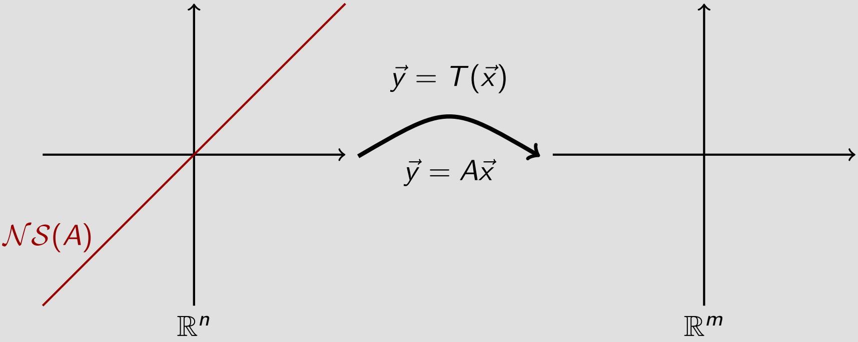

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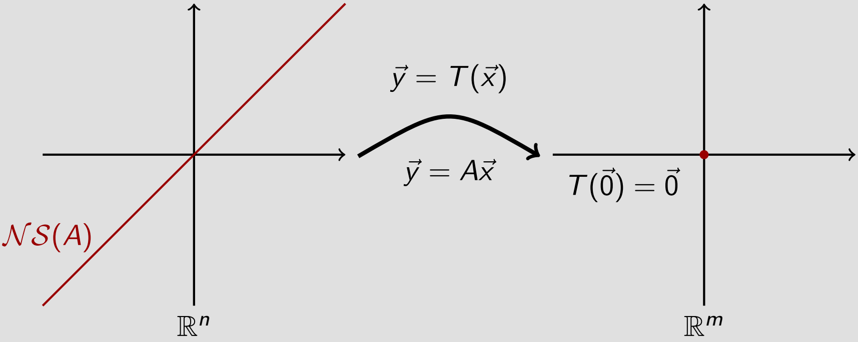

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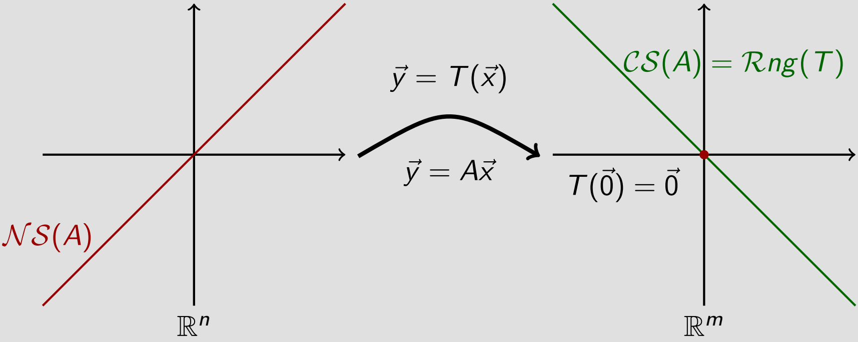

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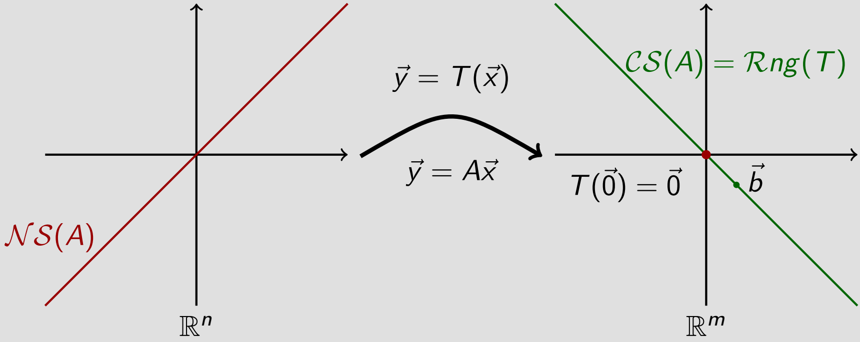

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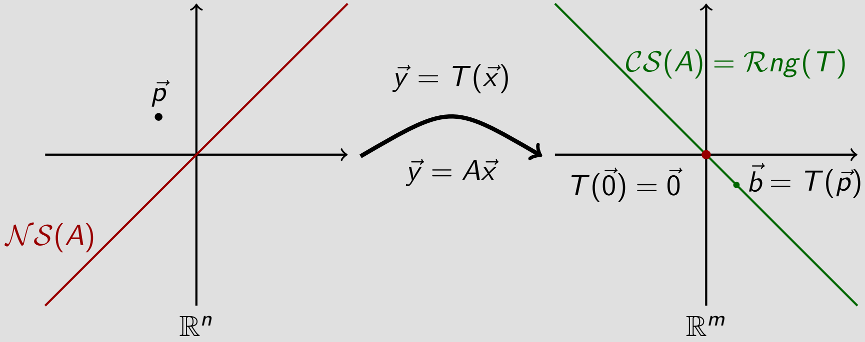

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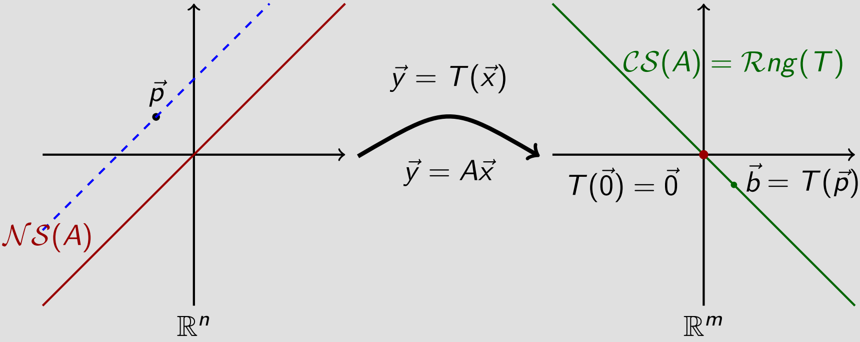

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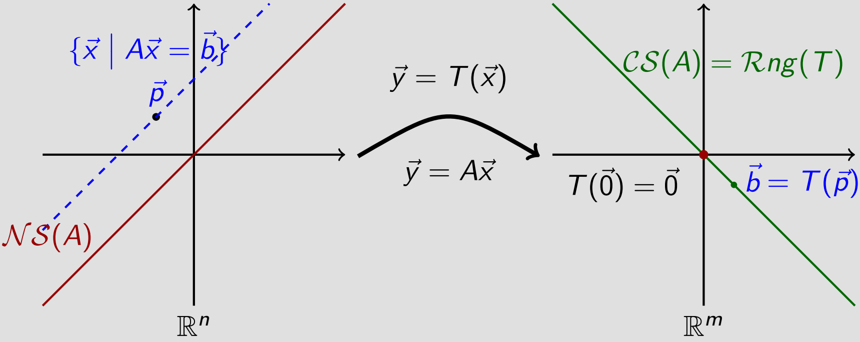

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So, “find” means to find a *linearly independent spanning* set.

Example—Null Space and Column Space

Find the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix}$$

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Basic Fact about Vector SubSpaces

Let $\mathbb{V}$ be a vector subspace. Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ are in $\mathbb{V}$.
Then each vector in $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ lies in $\mathbb{V}$.

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$\mathbb{V}$ is a vector subspace of $\mathbb{R}^3$ if and only if

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- $\mathbb{V} = \mathbb{R}^3$, or
- $\mathbb{V}$ is a line thru $\vec{0}$, or
- $\mathbb{V}$ is a plane thru $\vec{0}$.

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- $\mathbb{V}$ is a k-plane thru $\vec{0}$ (for some $1 < k < n$).