Subspaces of Euclidean Space $\mathbb{R}^n$
How do we recognize these?

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Vector Subspaces—Basic Example & Basic Fact

A collection $\mathbb{V}$ of vectors (in $\mathbb{R}^n$) is a *vector subspace* (of $\mathbb{R}^n$) if and only if

- $\vec{0}$ is in $\mathbb{V}$,
- $\mathbb{V}$ *closed with respect to vector addition* ($\vec{u}, \vec{v}$ in $\mathbb{V} \implies \vec{u} + \vec{v}$ in $\mathbb{V}$)
- $\mathbb{V}$ *closed with respect to scalar mult* (s scalar , $\vec{v}$ in $\mathbb{V} \implies s\vec{v}$ in $\mathbb{V}$)

Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a vector subspace.

Basic Fact about Vector SubSpaces

Let $\mathbb{V}$ be a vector subspace. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are in $\mathbb{V}$.
Then each vector in $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ lies in $\mathbb{V}$.

Possible Vector Subspaces of $\mathbb{R}^n$

$\mathbb{V}$ is a vector subspace of $\mathbb{R}^n$ if and only if

- $\mathbb{V} = \{\vec{0}\}$, or
- $\mathbb{V} = \mathbb{R}^n$, or
- $\mathbb{V}$ is a line thru $\vec{0}$, or
- $\mathbb{V}$ is a 2-plane thru $\vec{0}$, or
- $\mathbb{V}$ is a k-plane thru $\vec{0}$ (for some $1 < k < n$).

Why does the above list include **all** vector subspaces of $\mathbb{R}^n$?

Possible Vector Subspaces of $\mathbb{R}^2$

$\mathbb{V}$ is a vector subspace of $\mathbb{R}^2$ if and only if

(a) $\mathbb{V} = \{\vec{0}\}$, or

(b) $\mathbb{V} = \mathbb{R}^2$, or

(c) $\mathbb{V}$ is a line thru $\vec{0}$.

Let's see why this is true. Let $\mathbb{V}$ be a vector subspace of $\mathbb{R}^2$.

If $\vec{0}$ is the *only* vector in $\mathbb{V}$, then $\mathbb{V} = \{\vec{0}\}$ so (a) holds and we are done.

Assume there is a non-zero vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{V}$; so either $a \neq 0$ or $b \neq 0$, and we may assume $a \neq 0$.

Since $\mathbb{V}$ is closed wrt scalar multiplication, the line $\mathbb{L} = \mathcal{S}pan\{\vec{v}\}$ lies in $\mathbb{V}$.
If $\mathbb{V} = \mathbb{L}$, then (c) holds and we are done.

Thus we may assume there is a vector $\vec{w} = \begin{bmatrix} c \\ d \end{bmatrix}$ in $\mathbb{V}$ that is not in $\mathbb{L}$.

We claim that $\mathbb{V} = \mathbb{R}^2$, so (b) holds.

Why must $\mathbb{V} = \mathbb{R}^2$?

$\mathbb{V}$ is a vector subspace of $\mathbb{R}^2$ with $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} c \\ d \end{bmatrix}$ vectors in $\mathbb{V}$.

The Basic Fact about VSS says that every vector in $\mathcal{S}pan\{\vec{v}, \vec{w}\}$ belongs to $\mathbb{V}$. Now we explain why $\mathcal{S}pan\{\vec{v}, \vec{w}\} = \mathbb{R}^2$, which tells us that $\mathbb{V} = \mathbb{R}^2$.

We know that $a \neq 0$ and $\vec{w}$ is not in $\mathbb{L} = \mathcal{S}pan\{\vec{v}\}$. This means that for any scalar s, $\vec{w} \neq s\vec{v}$, right?

It follows that

$$\begin{bmatrix} c \\ d \end{bmatrix} = \vec{w} \neq \frac{c}{a}\,\vec{w} = \frac{c}{a}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ bc/a \end{bmatrix}$$

and therefore $\boxed{d \neq bc/a}$.

Let $\vec{z}$ be a vector in $\mathbb{R}^2$. We show that we can solve $x\vec{v} + y\vec{w} = \vec{z}$.

The coefficient matrix for this vector equation is just $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Why can we solve $x\vec{v} + y\vec{w} = \vec{z}$?

The coefficient matrix for this vector equation is just $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Also, we know that $a \neq 0$ and $\boxed{d \neq bc/a}$.

Let's apply two elementary row ops to the above coefficient matrix. We get

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \xrightarrow[R_2 - b*R_1]{\frac{1}{a}*R_1} \begin{bmatrix} 1 & \dfrac{c}{a} \\ 0 & d - \dfrac{bc}{a} \end{bmatrix}$$

Notice that $d - bc/a \neq 0$, so both rows of the above REF have row leaders. This means that we can solve the equation

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{z}$$

for **any** rhs vector $\vec{z}$. Thus every $\vec{z}$ in $\mathbb{R}^2$ is an LC of $\vec{v}$ and $\vec{w}$. This means that $\mathcal{S}pan\{\vec{v}, \vec{w}\} = \mathbb{R}^2$, so $\mathbb{V} = \mathbb{R}^2$.