Subspaces of Euclidean Space $\mathbb{R}^n$

Linear Algebra
MATH 2076

University of Cincinnati

Vector Subspaces—Basic Example

$\mathbb{V}$ is a *vector subspace* (of $\mathbb{R}^n$) if and only if

- $\vec{0}$ is in $\mathbb{V}$,
- $\mathbb{V}$ *closed with respect to vector addition* ($\vec{u}, \vec{v}$ in $\mathbb{V} \implies \vec{u} + \vec{v}$ in $\mathbb{V}$)
- $\mathbb{V}$ *closed with respect to scalar mult* (s scalar , $\vec{v}$ in $\mathbb{V} \implies s\vec{v}$ in $\mathbb{V}$)

Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a vector subspace.

Column Space of a Matrix

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix; so, each $\vec{a}_j$ is in $\mathbb{R}^m$.

The *column space* $\mathcal{CS}(A)$ of A is the span of the columns of A, i.e., $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$.

Three Ways to View $\mathcal{CS}(A)$

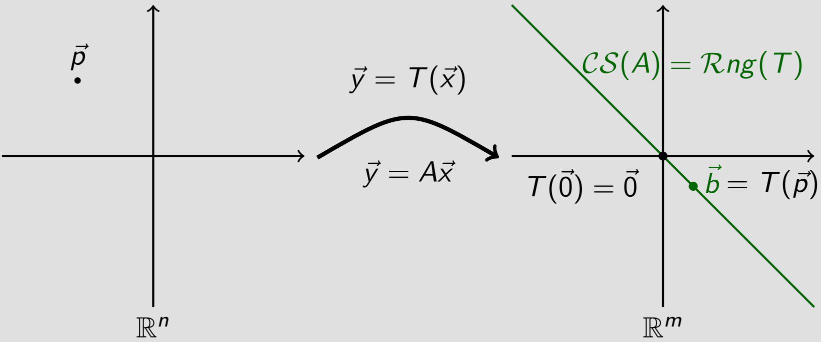
The column space $\mathcal{CS}(A)$ of $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ is:

- $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$
- $\mathcal{CS}(A) = \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\}$
- $\mathcal{CS}(A) = \mathcal{R}ng(T)$ where $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

Three Ways to View the Column Space $\mathcal{CS}(A)$

The column space $\mathcal{CS}(A)$ of $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n\end{bmatrix}$ is:

- $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$
- $\mathcal{CS}(A) = \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\}$
- $\mathcal{CS}(A) = \mathcal{R}ng(T)$ where $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

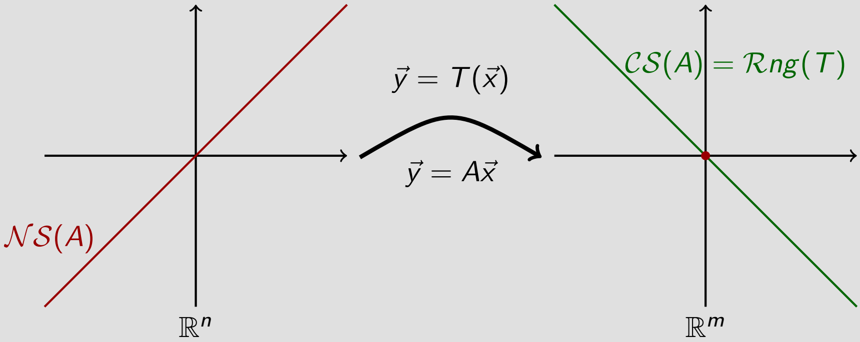


Vector Subspaces—Another Basic Example

Again, let A be an $m \times n$ matrix. The *null space* $\mathcal{NS}(A)$ of A is

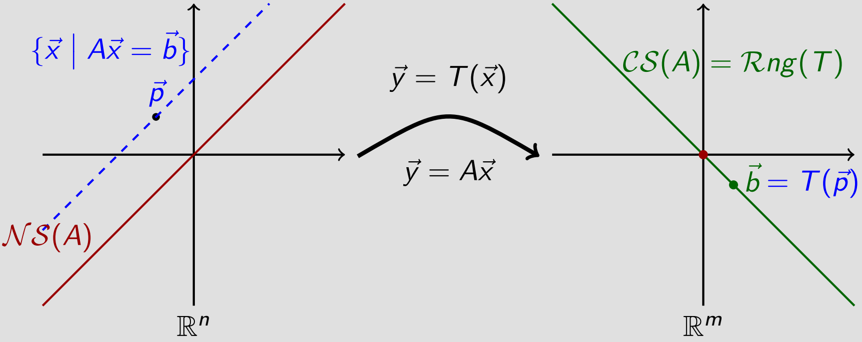
$$\mathcal{NS}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\};$$

just the solution set for the homogeneous equation $A\vec{x} = \vec{0}$. This is a vector subspace of $\mathbb{R}^n$.



$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ an $m \times n$ matrix and $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

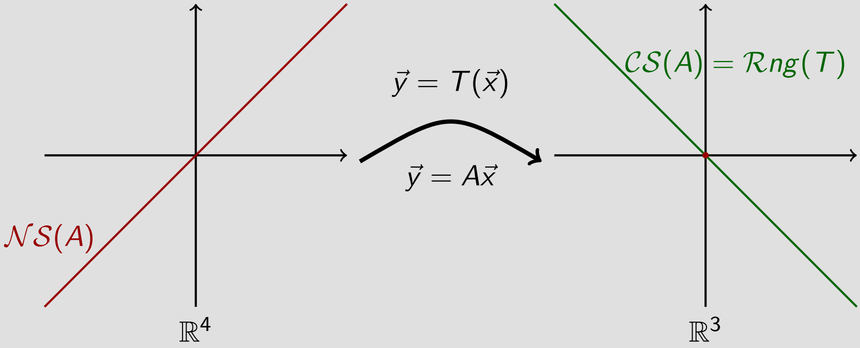
$$\begin{aligned} \mathcal{NS}(A) &= \{\vec{x} \mid A\vec{x} = \vec{0}\} \quad \text{and} \\ \mathcal{CS}(A) &= \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \\ &= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\} \\ &= \mathcal{CS}(A) = \mathcal{R}ng(T) \end{aligned}$$



Null Space and Column Space Example

Find the null space and column space of

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix}$$



What should we do now? How about row reducing A?

"Finding" Null Space and Column Space

To "find" the null space $\mathcal{NS}(A)$ and column space $\mathcal{CS}(A)$ of a matrix A:

- row reduce A to E, a REF (or RREF) for A
- columns of E containing row leaders correspond to *pivot* columns of A
- the *pivot* columns of A are LI and span $\mathcal{CS}(A)$
- write the SS for $A\vec{x} = \vec{0}$ in parametric vector form
- identify LI vectors that span $\mathcal{NS}(A)$

So, "find" means to find a *linearly independent spanning* set.

Example—Null Space and Column Space

Find the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix}$$