Subspaces of Euclidean Space $\mathbb{R}^n$

Linear Algebra
MATH 2076

What is a subspace?

Let $\mathbb{V}$ be a collection of vectors in $\mathbb{R}^n$. (For example, $\mathbb{V}$ could be a solution set to some equation, or it could be all the vectors that have third coordinate -7.)

We say that $\mathbb{V}$ *closed with respect to scalar multiplication* if and only if whenever $\vec{v}$ is in $\mathbb{V}$ and s is any scalar, then $s\vec{v}$ is also in $\mathbb{V}$. For example, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}\}$ (for some $\vec{v}$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to scalar multiplication. In fact, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ (for any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to scalar multiplication.

We say that $\mathbb{V}$ *closed with respect to vector addition* if and only if whenever $\vec{u}$ and $\vec{v}$ are in $\mathbb{V}$, then $\vec{u} + \vec{v}$ is also in $\mathbb{V}$. For example, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}\}$ (for some $\vec{v}$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to vector addition. In fact, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ (for any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to vector addition.

We call $\mathbb{V}$ a *vector subspace* of $\mathbb{R}^n$ if and only if ...

What is a subspace?

Let $\mathbb{V}$ be a collection of vectors in $\mathbb{R}^n$.
We call $\mathbb{V}$ a *vector subspace* of $\mathbb{R}^n$ if and only if

- $\vec{0}$ is in $\mathbb{V}$
- $\mathbb{V}$ *closed with respect to vector addition* ($\vec{u}, \vec{v}$ in $\mathbb{V} \implies \vec{u} + \vec{v}$ in $\mathbb{V}$)
- $\mathbb{V}$ *closed with respect to scalar mult* (s scalar , $\vec{v}$ in $\mathbb{V} \implies s\vec{v}$ in $\mathbb{V}$)

Some simple examples:

- $\mathbb{V} = \{\vec{0}\}$ is the *trivial* vector subspace
- $\mathbb{V} = \mathbb{R}^n$ is a vector subspace of itself (also kinda *trivial*)
- $\mathbb{V} = \mathcal{S}pan\{\vec{v}\}$ (for any $\vec{v}$ in $\mathbb{R}^n$)
- $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ (for any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$)

A simple non-example:

- $\mathbb{V} = \Big\{$all $\vec{v}$ in $\mathbb{R}^4$ with third coordinate -7$\Big\}$ is not a subspace

More Examples—Which are, or are not, vector subspaces?

For each $\mathbb{V}$, decide whether or not $\mathbb{V}$ is closed with respect to scalar multiplication and/or closed with respect to vector addition.

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x + y = 1 \right\}$$

$\vec{0}$ not in $\mathbb{V}$, so not VSS

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; xy \geq 0 \right\}$$

$\mathbb{V}$ not closed wrt vector add

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x \geq 0 \text{ and } y \geq 0 \right\}$$

$\mathbb{V}$ not closed wrt scalar mult

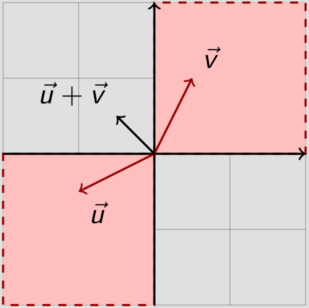

Figure: $xy \geq 0$

Vector Subspaces—Basic Example

Recall that a collection $\mathbb{V}$ of vectors (in $\mathbb{R}^n$) is a *vector subspace* (of $\mathbb{R}^n$) if and only if

- $\vec{0}$ is in $\mathbb{V}$,
- $\mathbb{V}$ *closed with respect to vector addition* ($\vec{u}, \vec{v}$ in $\mathbb{V} \implies \vec{u} + \vec{v}$ in $\mathbb{V}$)
- $\mathbb{V}$ *closed with respect to scalar mult* (s scalar, $\vec{v}$ in $\mathbb{V} \implies s\vec{v}$ in $\mathbb{V}$)

Let $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$. Let's show that $\mathbb{V}$ is closed wrt vector addition. Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{V}$. This means there are scalars $s_1, s_2, \ldots, s_p$ and $t_1, t_2, \ldots, t_p$ with

$$\vec{u} = s_1\vec{v}_1 + \cdots + s_p\vec{v}_p \quad \text{and} \quad \vec{v} = t_1\vec{v}_1 + \cdots + t_p\vec{v}_p$$

so

$$\vec{u} + \vec{v} = (s_1 + t_1)\vec{v}_1 + (s_2 + t_2)\vec{v}_2 + \cdots + (s_p + t_p)\vec{v}_p$$

which is a vector in $\mathbb{V}$.

Homework: Show that $\mathbb{V}$ is closed wrt scalar multiplication.

Vector Subspaces—Basic Example

Just saw that any $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is both closed wrt vector addition and closed wrt scalar multiplication.

Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a vector subspace.

In fact, *every* vector subspace can be expressed this way!

Example (Column Space of a Matrix)

The *column space* $\mathcal{CS}(A)$ of a matrix A is the span of the columns of A. Thus is A is an $m \times n$ matrix, then $\mathcal{CS}(A)$ is a VSS of $\mathbb{R}^m$.

If $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix}$, then $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.