The Invertible Matrix Theorem

Linear Algebra
MATH 2076

Invertible Matrices

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When does $A\vec{x} = \vec{b}$ have a solution for *every* rhs $\vec{b}$?

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$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

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What can we say?

Solving $A\vec{x} = \vec{b}$

First, do elem row ops to get $\left[A \,|\, \vec{b}\right] \xrightarrow[\text{to REF}]{\text{row reduce}} \left[E \,|\, \vec{c}\right]$.

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Solving $A\vec{x} = \vec{b}$

First, do elem row ops to get $\left[A \mid \vec{b}\right] \xrightarrow[\text{to REF}]{\text{row reduce}} \left[E \mid \vec{c}\right]$.

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Two possibilities:

Some free variables—get **infinitely many** solutions.

No free variables—get a **unique** solution.

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What's important:

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What's important: any one of last 5 statements true $\implies$ A is invertible.

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What happens if we do T then S, or, S then T?

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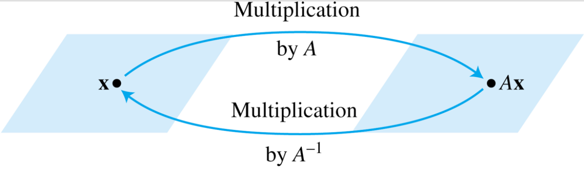
Can "see" this too!



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

Invertible Matrices and Linear Transformations

Let A be an $n \times n$ matrix. Define a LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.
Suppose A is invertible. What does this tell us about T?

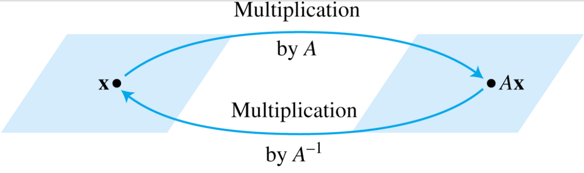
Since A is invertible, have A^{-1}, so get LT $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^n$, $S(\vec{y}) = A^{-1}\vec{y}$.

What happens if we do T then S, or, S then T?

Look at $S\big(T(\vec{x})\big) = S\big(A\vec{x}\big) = A^{-1}A\vec{x} = I\vec{x} = \vec{x}$.

Similarly, $T\big(S(\vec{y})\big) = T\big(A^{-1}\vec{y}\big) = A\,A^{-1}\vec{y} = I\vec{y} = \vec{y}$.

Can "see" this too!



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Above says $T^{-1} = S$.

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If A is square and invertible, then so is A^T and $\boxed{(A^T)^{-1} = \left(A^{-1}\right)^T}$.