The Invertible Matrix Theorem

Linear Algebra
MATH 2076
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When this holds, there is only \textbf{one} such matrix $C$; we call it $A^{-1}$. 
An \( n \times n \) matrix \( A \) is **invertible** if and only if there is another \( n \times n \) matrix \( C \) with \( A \cdot C = I = C \cdot A \).

When this holds, there is only one such matrix \( C \); we call it \( A^{-1} \).

Look at super-sized augmented matrix \( \begin{bmatrix} A & \vdots & I \end{bmatrix} \).
Invertible Matrices

An $n \times n$ matrix $A$ is *invertible* if and only if there is another $n \times n$ matrix $C$ with $A C = I = C A$.

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Look at super-sized augmented matrix $\begin{bmatrix} A & \vdots & I \end{bmatrix}$. Put into reduced REF.
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An $n \times n$ matrix $A$ is invertible if and only if there is another $n \times n$ matrix $C$ with $A C = I = C A$.

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Look at super-sized augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. Put into reduced REF.

Do elementary row operations to get $\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{row reduce to}} \begin{bmatrix} E & F \end{bmatrix}$. 
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Look at super-sized augmented matrix \( [A : I] \). Put into \textit{reduced} REF.

Do elementary row operations to get \( \begin{bmatrix} A : I \end{bmatrix} \xrightarrow{\text{row reduce to reduced REF}} \begin{bmatrix} E : F \end{bmatrix} \).

Get two possibilities:
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Do elementary row operations to get $[A : I] \xrightarrow{\text{row reduce to reduced REF}} [E : F]$.

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If \( E = I \), then \( F = A^{-1} \).
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Properties of Invertible Matrices

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When does $A\vec{x} = \vec{0}$ have a unique solution?

When does $A\vec{x} = \vec{b}$ have a solution for every rhs $\vec{b}$?
Example

All questions about solns to SLEs (or VEs or MEs and more!) can be answered by looking at a REF of appropriate matrix.
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Suppose the augmented matrix for some SLE has the following REF.

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\begin{bmatrix}
1 & 2 & 3 & 4 & 3 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
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What can we say?
Solving $A\vec{x} = \vec{b}$

First, do elem row ops to get $[A | \vec{b}] \xrightarrow{\text{row reduce}} [E | \vec{c}]$. If $\vec{c}$ has a row leader, there are NO solutions; assume otherwise. Identify the columns of $E$ that do not have row leaders; the corresponding variables are free. Two possibilities:

- Some free variables — get infinitely many solutions.
- No free variables — get a unique solution.
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First, do elem row ops to get $[A \mid \vec{b}] \xrightarrow{\text{row reduce to REF}} [E \mid \vec{c}]$.

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(If one statement holds, all do; if one statement is false, all are false.)

- $A$ is invertible.
- $[A : I]$ row reduce to reduced REF $[E : F]$
- $[A : I]$ row equivalent to $I$
- $A \vec{x} = \vec{0}$ has no non-zero solutions.
- The columns of $A$ are linearly independent.
- $A \vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^n$.
- The columns of $A$ span all of $\mathbb{R}^n$. 

What's important: any one of last 5 statements true $\Rightarrow$ $A$ is invertible.
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The Invertible Matrix Theorem—a small part

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What’s important: any one of last 5 statements true $\implies A$ is invertible.
Let $A$ be an $n \times n$ matrix.
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What happens if we do $T$ then $S$, or, $S$ then $T$?

Look at $S(T(\vec{x})) = S(A\vec{x}) = A^{-1}A\vec{x} = I\vec{x} = \vec{x}$.

Similarly, $T(S(\vec{y})) = T(A^{-1}\vec{y}) = AA^{-1}\vec{y} = I\vec{y} = \vec{y}$.

Can “see” this too!

Above says $T^{-1} = S$. 

Section 2.3
Invertible Matrices
Let $A$ be an $n \times n$ matrix. Define a LT $\mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$. Suppose $A$ is invertible.
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Since $A$ is invertible, have $A^{-1}$, so get LT $\mathbb{R}^n \rightarrow \mathbb{R}^n$, $S(\vec{y}) = A^{-1}\vec{y}$. 

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Let \( A \) be an \( n \times n \) matrix. Define a LT \( \mathbb{R}^n \to \mathbb{R}^n \) by \( T(\vec{x}) = A\vec{x} \).

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Since \( A \) is invertible, have \( A^{-1} \), so get LT \( \mathbb{R}^n \to \mathbb{R}^n \), \( S(\vec{y}) = A^{-1}\vec{y} \).

What happens if we do \( T \) then \( S \), or, \( S \) then \( T \)?

Look at \( S(T(\vec{x})) = S(A\vec{x}) = A^{-1}A\vec{x} \)
Invertible Matrices and Linear Transformations

Let $A$ be an $n \times n$ matrix. Define a LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

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Since \( A \) is invertible, have \( A^{-1} \), so get LT \( \mathbb{R}^n \xrightarrow{S} \mathbb{R}^n \), \( S(\vec{y}) = A^{-1}\vec{y} \).

What happens if we do \( T \) then \( S \), or, \( S \) then \( T \)?

Look at \( S(T(\vec{x})) = S(A\vec{x}) = A^{-1}A\vec{x} = I\vec{x} = \vec{x} \).

Similarly, \( T(S(\vec{y})) = T(A^{-1}\vec{y}) = AA^{-1}\vec{y} = I\vec{y} = \vec{y} \).

Can “see” this too!

\[ A^{-1} \text{ transforms } Ax \text{ back to } x. \]
Let $A$ be an $n \times n$ matrix. Define a LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

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Can “see” this too!

Above says $T^{-1} = S$.

$A^{-1}$ transforms $Ax$ back to $x$. 
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The \textit{transpose} $A^T$ of a matrix $A$ is given by “reflecting $A$ across its main diagonal”.

For example, $egin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

If $A$ is square and invertible, then so is $A^T$ and $(A^T)^{-1} = (A^{-1})^T$. 

Section 2.3

Invertible Matrices

3 February 2017
The transpose $A^T$ of a matrix $A$ is given by “reflecting $A$ across its main diagonal”. The rows (columns) of $A$ become the columns (rows) of $A^T$.

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Invertible Matrices and Transpose

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