The Invertible Matrix Theorem

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Invertible Matrices

An $n \times n$ matrix A is *invertible* if and only if there is another $n \times n$ matrix C with $\boxed{A\,C = I = C\,A}$.

When this holds, there is only **one** such matrix C; we call it A^{-1}.

Look at super-sized augmented matrix $\left[A \,\vdots\, I\right]$. Put into *reduced* REF.

Do elementary row operations to get $\left[A \,\vdots\, I\right] \xrightarrow[\text{reduced REF}]{\text{row reduce to}} \left[E \,\vdots\, F\right]$.

Get two possibilities:

If $E \neq I$, then A not invertible.

If $E = I$, then $F = A^{-1}$.

Properties of Invertible Matrices

Invertible matrices possess a bewildering number of characteristic properties. Our text book lists 26 different ways to see that a square matrix is invertible! See pp. 114, 116, 158, 173, 237, 423.

Here we focus on just a few of these. Let A be a square matrix. Consider the matrix equation $A\vec{x} = \vec{b}$.

When does $A\vec{x} = \vec{0}$ have a *unique* solution?

When does $A\vec{x} = \vec{b}$ have a solution for *every* rhs $\vec{b}$?

Example

All questions about solns to SLEs (or VEs or MEs and more!) can be answered by looking at a REF of appropriate matrix.

Suppose the augmented matrix for some SLE has the following REF.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

What can we say?

Solving $A\vec{x} = \vec{b}$

First, do elem row ops to get $\left[A \,|\, \vec{b}\right] \xrightarrow[\text{to REF}]{\text{row reduce}} \left[E \,|\, \vec{c}\right]$.

If $\vec{c}$ has a row leader, there are NO solutions; assume otherwise.

Identify the columns of E that do *not* have row leaders;
the corresponding variables are *free*.

Two possibilities:

Some free variables—get **infinitely many** solutions.

No free variables—get a **unique** solution.

The Invertible Matrix Theorem—a small part

For an $n \times n$ matrix A, the following statements are equivalent.
(If one statement holds, all do; if one statement is false, all are false.)

- A is invertible. $\quad [A \,\vdots\, I] \xrightarrow[\text{reduced REF}]{\text{row reduce to}} [E \,\vdots\, F]$
- A is row equivalent to I. $\quad E = I$
- $A\vec{x} = \vec{0}$ has no non-zero solutions. $\quad$ No free variables!
  (When $A\vec{x} = \vec{b}$ has a soln, it is unique.)
- The columns of A are linearly independent.
- $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^n$.
- The columns of A span all of $\mathbb{R}^n$.

What's important: any one of last 5 statements true $\Longrightarrow$ A is invertible.

Invertible Matrices and Linear Transformations

Let A be an $n \times n$ matrix. Define a LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.
Suppose A is invertible. What does this tell us about T?

Since A is invertible, have A^{-1}, so get LT $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^n$, $S(\vec{y}) = A^{-1}\vec{y}$.

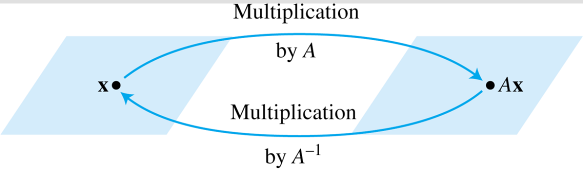
What happens if we do T then S, or, S then T?

Look at $S\big(T(\vec{x})\big) = S\big(A\vec{x}\big) = A^{-1}A\vec{x} = I\vec{x} = \vec{x}$.

Similarly, $T\big(S(\vec{y})\big) = T\big(A^{-1}\vec{y}\big) = A\,A^{-1}\vec{y} = I\vec{y} = \vec{y}$.

Can "see" this too!

Above says $T^{-1} = S$.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

Invertible Matrices and Products

Let A and B be invertible $n \times n$ matrices.

Not hard to see that AB is invertible.

In fact, $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$.

This because

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I\,B = B^{-1}B = I.$$

Invertible Matrices and Transpose

The *transpose* A^T of a matrix A is given by "reflecting A across its main diagonal". The rows (columns) of A become the columns (rows) of A^T.

More precisely, if $A = [a_{ij}]$, then $A^T = [a_{ji}]$.

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

If A is square and invertible, then so is A^T and $\boxed{(A^T)^{-1} = \left(A^{-1}\right)^T}$.