The Inverse of a Matrix

Linear Algebra MATH 2076



An $n \times n$ matrix A is *invertible* if and only if there is another $n \times n$ matrix C with A = C = C = C = A, where C = C = A, where C = A = A

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How can we tell when a matrix is invertible?

How can we find such a matrix C?



An $n \times n$ matrix A is *invertible* if and only if there is another $n \times n$ matrix C with A = C = C A.

When this holds, there is only ONE such matrix C.

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Remember, not all matrices have an inverse.



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So, above is an example of an A with $A^{-1} = A$, and $A \neq \pm I$.



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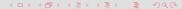
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First,
$$I = \begin{bmatrix} \vec{e_1} & \vec{e_2} \dots \vec{e_n} \end{bmatrix}$$
, where $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, ..., $\vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

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so
$$\operatorname{Col}_j(I) = \vec{e_j} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$
 where the one 1 appears in the j^{th} row.

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Look at super-sized augmented matrix [A : I].

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Do elem row ops to get $[A : I] \xrightarrow{\text{row reduce to} \\ \text{reduced REF}} [E : F].$

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If E = I, then $F = A^{-1}$.

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Look at super-sized augmented matrix A : I. Put into *reduced* REF.

Do elementary row operations to get $\begin{bmatrix} A \\ \vdots \end{bmatrix} \xrightarrow{\text{row reduce to} \atop \text{reduced REF}} \begin{bmatrix} E \\ \vdots \end{bmatrix} F$.

Get two possibilities:

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If
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If $E \neq I$, then A not invertible.

If E = I, then $F = A^{-1}$.

Determine if
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
 is invertible, and find A^{-1} if it exists.

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Gotta row reduce

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

 $\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

ı	Г1	2	3	4	1	0	0	0		Г1	2	3	4	1	0	0	٦٥	
					0								3					
	1	1	1	2	0	0	1	0	R_4-R_3	1	1	1	2	0	0	1	0	
	1	1	1	1	0	0	0	1		0	0	0	-1	0	0	-1	1	

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

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 $\frac{R_3 - R_2}{-R_4} \xrightarrow{\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}} \xrightarrow{R_3 + R_4} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[0]{R_2-R_1} \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

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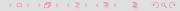
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\frac{R_1 - 2R_2}{-3R_3 - 4R4}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So,
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
 is invertible, and $A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

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This is not hard to do, right?



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