

# The Inverse of a Matrix

Linear Algebra  
MATH 2076



# Invertible Matrices

An  $n \times n$  matrix  $A$  is *invertible* if and only if there is another  $n \times n$  matrix  $C$  with  $AC = I = CA$ , where  $I$  is  $n \times n$  identity matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Some matrices are not invertible:  $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$  is not invertible.

How can we tell when a matrix is invertible?

How can we find such a matrix  $C$ ?

# Invertible Matrices

An  $n \times n$  matrix  $A$  is *invertible* if and only if there is another  $n \times n$  matrix  $C$  with  $AC = I = CA$ .

When this holds, there is only ONE such matrix  $C$ .

Suppose we have  $C_1A = I = AC_2$ . Then

$$C_1 = C_1I = C_1(AC_2) = (C_1A)C_2 = IC_2 = C_2$$

so  $C_1 = C_2$ .

We call  $C$  the *inverse* of  $A$  and write  $A^{-1} = C$ .

Remember, not all matrices have an inverse.

## Example

Remember, for numbers  $x^{-1} = x \implies x^2 = 1$ , so  $x = \pm 1$ .

Look at

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So, above is an example of an  $A$  with  $A^{-1} = A$ , and  $A \neq \pm I$ .

## Example

Look at

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so,

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$$

So, above is another example of an  $A$  with  $A^{-1} = A$ , and  $A \neq \pm I$ .

## The significance of $AC = I$ for a given $A$

Suppose there is an  $n \times n$  matrix  $C$  with  $AC = I$ .

Look at the equation  $A\vec{x} = \vec{b}$  for some given rhs  $\vec{b}$ .

Try  $\vec{x} = C\vec{b}$ . Get

$$A\vec{x} = AC\vec{b} = I\vec{b} = \vec{b}.$$

When there is  $C$  with  $AC = I$ , we *know* that for *any* rhs  $\vec{b}$ ,  $A\vec{x} = \vec{b}$  has a solution, namely  $\vec{x} = C\vec{b}$ . Solutions always exist for *any* rhs  $\vec{b}$ !

What does this say about a REF for  $A$ ?

What does this say about the columns of  $A$ ?

## The significance of $CA = I$ for a given $A$

Suppose there is an  $n \times n$  matrix  $C$  with  $CA = I$ .

Look at the equation  $A\vec{x} = \vec{b}$  for some given rhs  $\vec{b}$ .

Suppose that  $\vec{p}$  is some solution, so  $A\vec{p} = \vec{b}$ . Multiply both sides by  $C$  to get

$$\vec{p} = I\vec{p} = CA\vec{p} = C\vec{b}.$$

When there is  $C$  with  $CA = I$ , we *know* that  $A\vec{x} = \vec{b}$  has the *unique* solution,  $\vec{x} = C\vec{b}$ . If there is a solution (for some rhs  $\vec{b}$ ), then the solution is unique!

What does this say about a REF for  $A$ ?

What does this say about the columns of  $A$ ?

## Finding $C$ (i.e., finding $A^{-1}$ )

First,  $A$  may be non-invertible, right? There may be no such  $C$ .

Looking for  $C$  with  $AC = I$ . Think about a variable  $n \times n$  matrix  $X$ .

Trying to solve  $AX = I$ . Look at the columns of each side.

Next,  $\text{Col}_j(AX) = A \text{Col}_j(X)$ , so we're trying to solve

$$AX = I \quad \text{or} \quad \text{Col}_j(AX) = \text{Col}_j(I) \quad \text{or} \quad A\vec{x} = \vec{e}_j$$

where  $\vec{x} = \text{Col}_j(X)$ , and we solve for each  $1 \leq j \leq n$ .

Look at super-sized augmented matrix  $[A : I]$ . Put into *reduced* REF.

Do elem row ops to get  $[A : I] \xrightarrow[\text{reduced REF}]{\text{row reduce to}} [E : F]$ . Two possibilities:

If  $E \neq I$ , then  $A$  not invertible.

If  $E = I$ , then  $F = A^{-1}$ .



## Finding $A^{-1}$ , if it exists

Look at super-sized augmented matrix  $[A : I]$ . Put into *reduced* REF.

Do elementary row operations to get  $[A : I] \xrightarrow[\text{reduced REF}]{\text{row reduce to}} [E : F]$ .

Get two possibilities:

If  $E \neq I$ , then  $A$  not invertible.

If  $E = I$ , then  $F = A^{-1}$ .

## Example

Determine if  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$  is invertible, and find  $A^{-1}$  if it exists.

Gotta row reduce

$$\left[ \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] .$$

$$\left[ \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_4 - R_3} \left[ \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{array} \right]$$

$$\begin{array}{l} \xrightarrow{R_3 - R_2} \\ \xrightarrow{-R_4} \end{array} \left[ \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array} \right] \begin{array}{l} \xrightarrow{R_3 + R_4} \\ \xrightarrow{-R_3} \end{array} \left[ \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array} \right]$$

$$\xrightarrow{R_2 - R_1} \left[ \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array} \right] \begin{array}{l} \xrightarrow{R_2 + R_3 + R_4} \\ \xrightarrow{-R_2} \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array} \right] \begin{array}{l} \xrightarrow{R_1 - 2R_2} \\ \xrightarrow{-3R_3 - 4R_4} \end{array} \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array} \right]$$

## Example

$$\text{So, } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is invertible, and } A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Should **check** that

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is not hard to do, right?