Linear Transformations

Linear Algebra
MATH 2076

The Matrix Transformation $\vec{y} = A\vec{x}$

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We write $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ (and say, T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$), meaning that $\mathbb{R}^n$ is the *domain* of T and $\mathbb{R}^m$ the *codomain*.

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This is called the *linearity principle*.

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but this means $T(\vec{x}) = A\vec{x}$ where A is the matrix with columns $\vec{a}_j$.

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The *columns* of A are simply $\boxed{\vec{a}_j = T(\vec{e}_j)}$.

Rotations of the Plane $\mathbb{R}^2$

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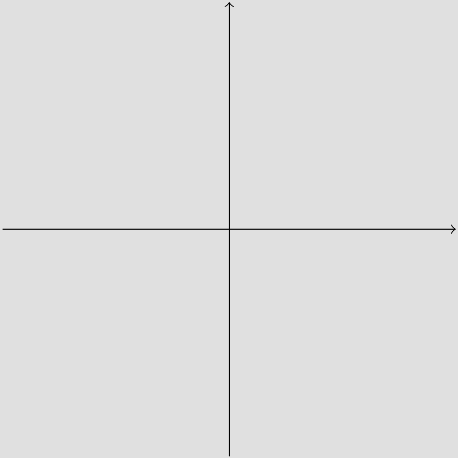

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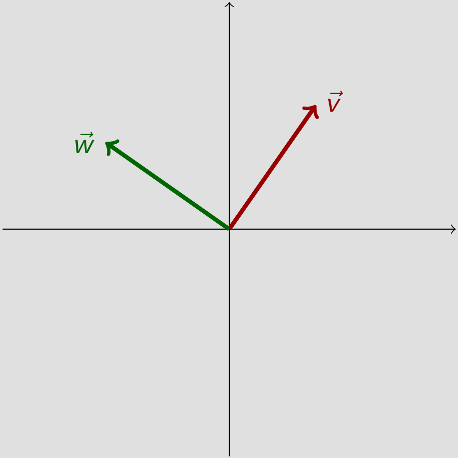

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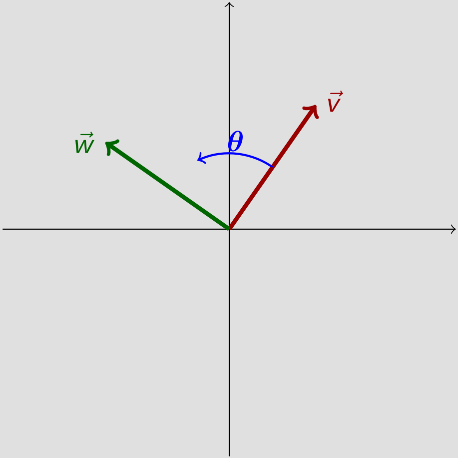

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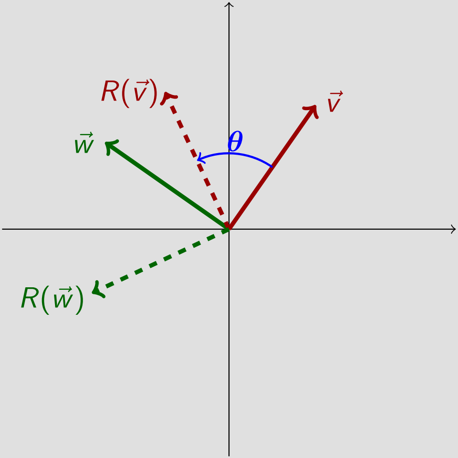

Figure: Rotating $\mathbb{R}^2$ by θ radians

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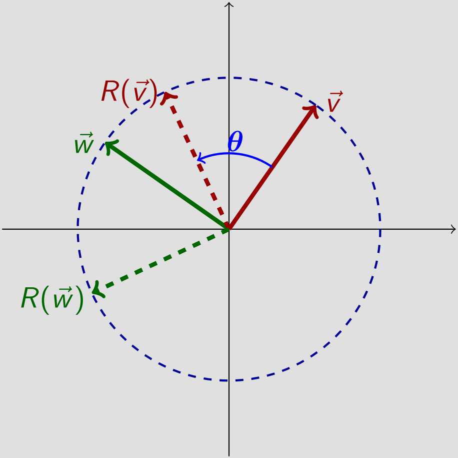
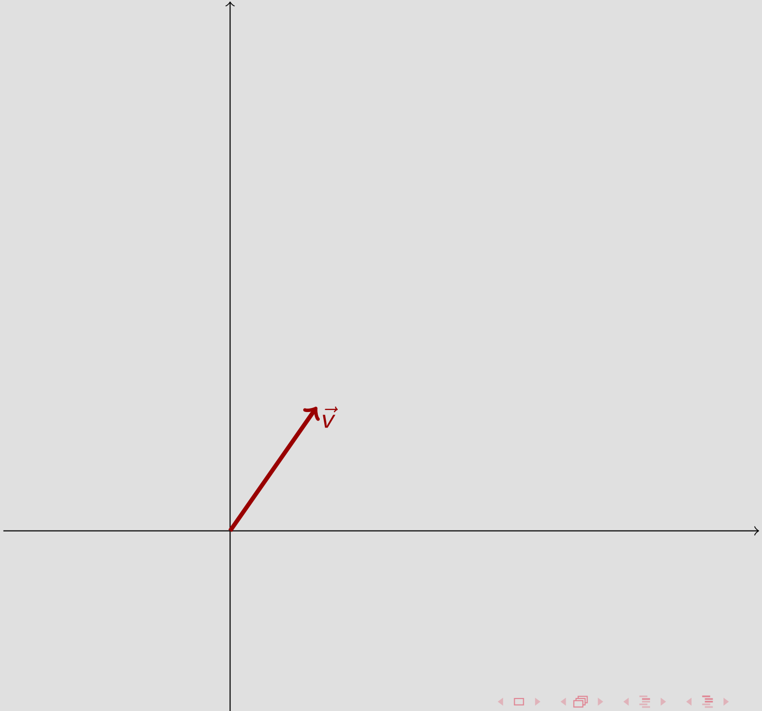
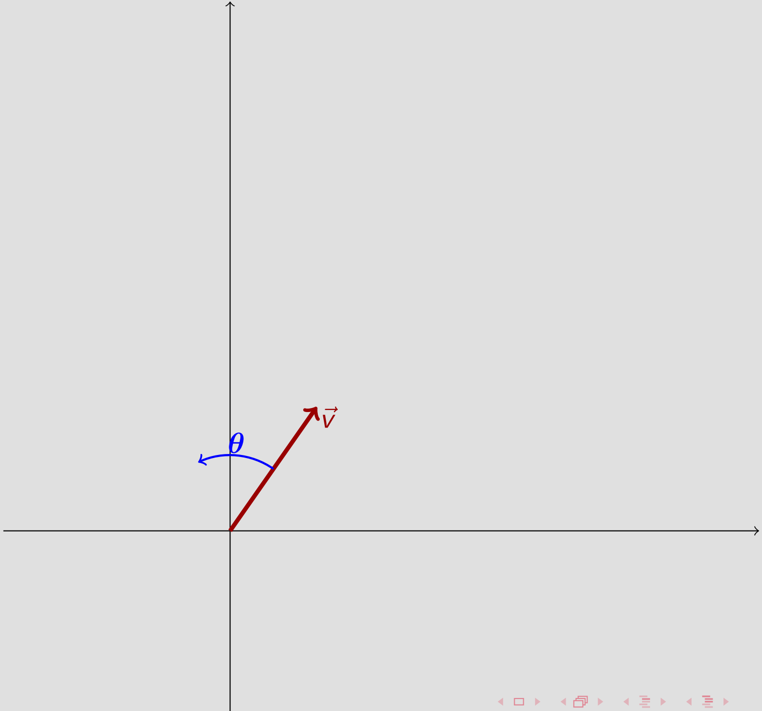
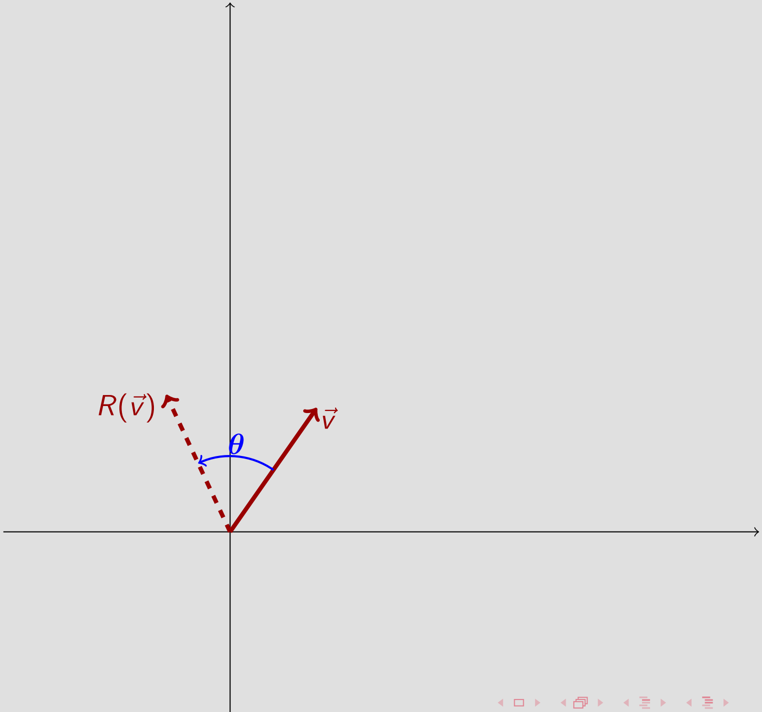
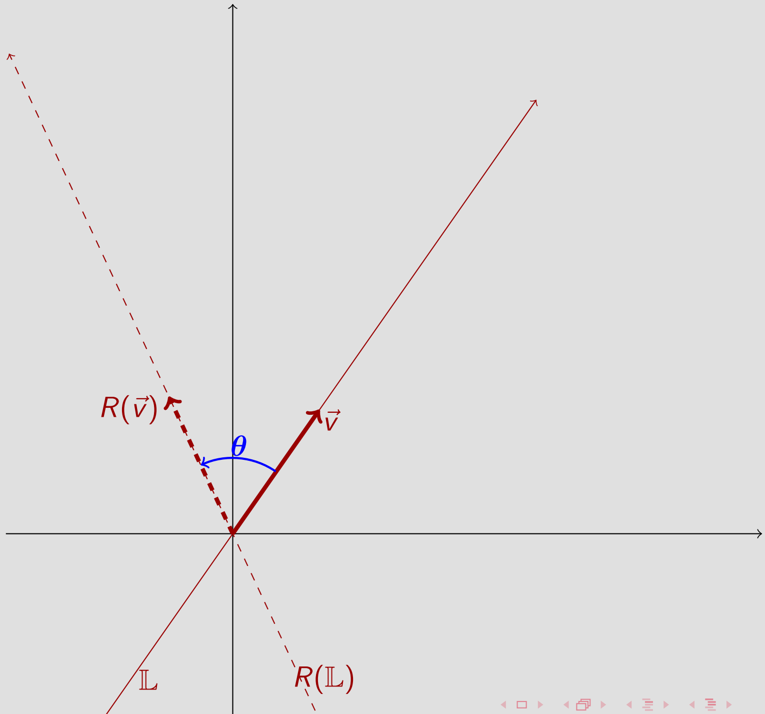


Figure: Rotating $\mathbb{R}^2$ by θ radians

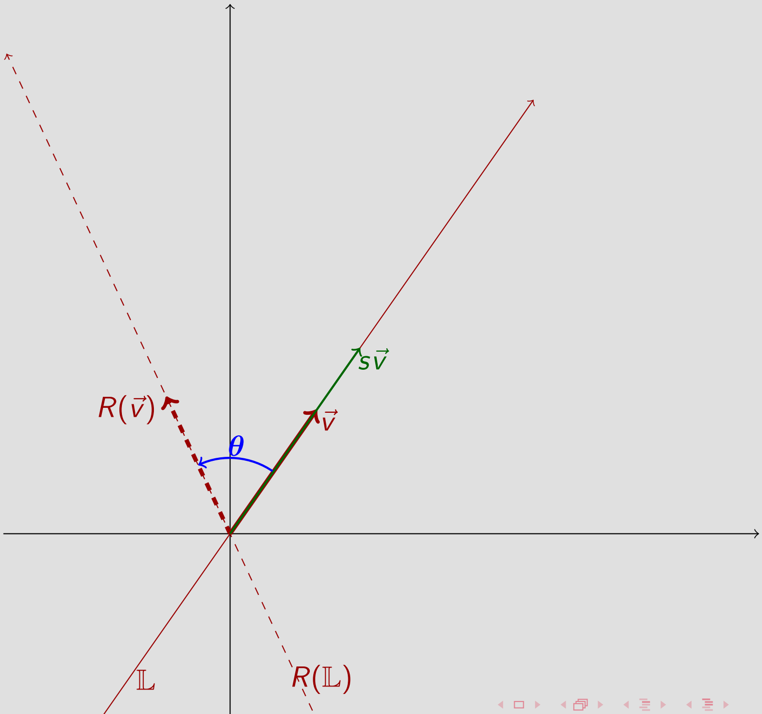

$\vec{v}$


θ
$\vec{v}$

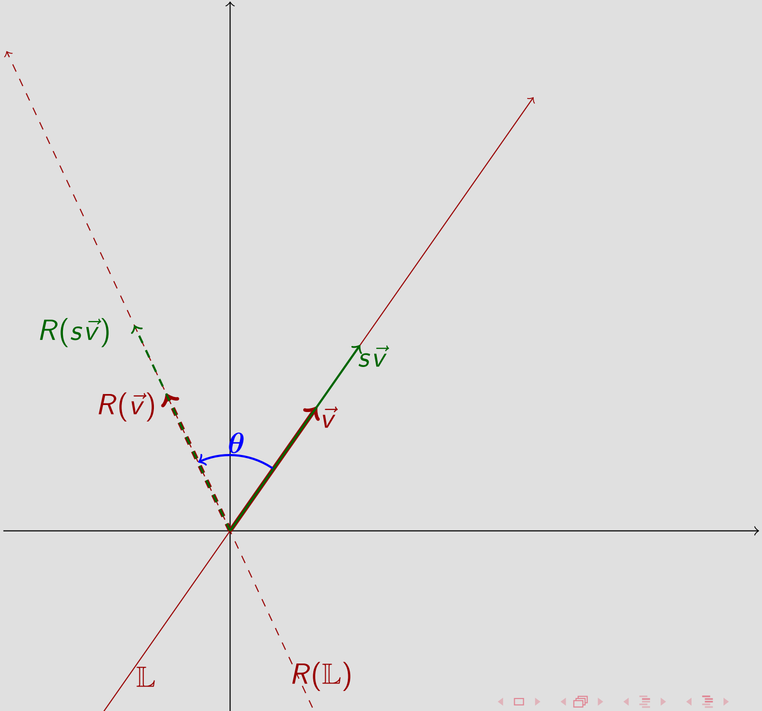

$R(\vec{v})$
θ
$\vec{v}$



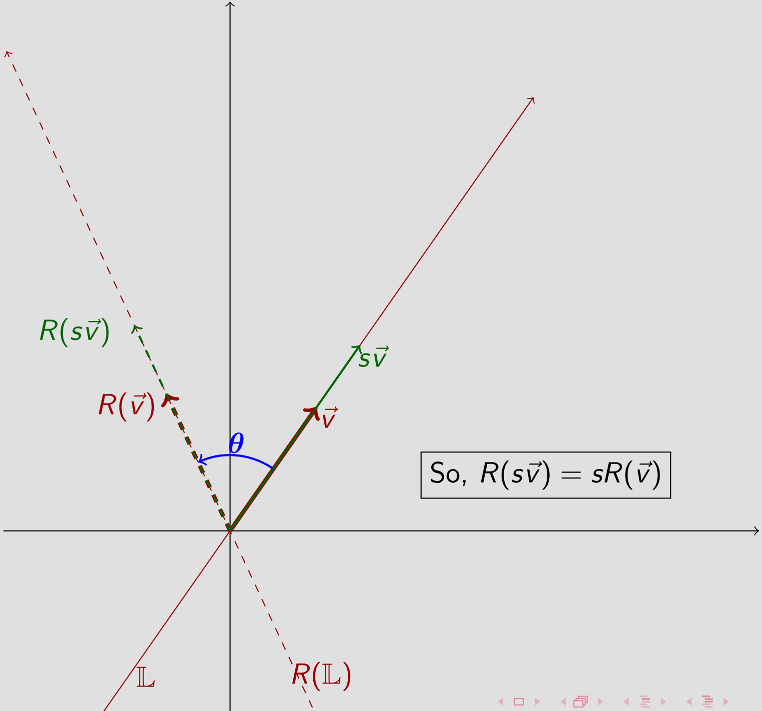
$R(\vec{v})$
$\vec{v}$
θ
$\mathbb{L}$
$R(\mathbb{L})$



$s\vec{v}$
$R(\vec{v})$
$\vec{v}$
θ
$\mathbb{L}$
$R(\mathbb{L})$



$R(s\vec{v})$
$s\vec{v}$
$R(\vec{v})$
$\vec{v}$
θ
$\mathbb{L}$
$R(\mathbb{L})$



So, $R(s\vec{v}) = sR(\vec{v})$