Matrix Transformations

Linear Algebra
MATH 2076

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In linear algebra, functions are usually called *transformations*. We write $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ to indicate that T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, meaning that the input variable $\vec{x}$ comes from $\mathbb{R}^n$ and the resulting output $\vec{y} = T(\vec{x})$ is a vector in $\mathbb{R}^m$.

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For each $\vec{x}$ in $\mathbb{R}^n$, $\vec{y} = T(\vec{x})$ is called the *image* of $\vec{x}$. If $\mathbb{S}$ is a bunch of vectors in $\mathbb{R}^n$ (i.e., $\mathbb{S} \subset \mathbb{R}^n$), then

$$T(\mathbb{S}) = \left\{ \text{all images } T(\vec{x}) \text{ where } \vec{x} \text{ is in } \mathbb{S} \right\}$$

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The *range* of T is exactly all rhs vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution.