Linear Independence

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Linear Combinations and Span

Suppose $s_1, s_2, \ldots, s_p$ are scalars and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are vectors (all in the same space $\mathbb{R}^n$). We call

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p$$

a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. We always have the *trivial* linear combination

$$0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p = \vec{0}.$$

Here we want to know when there is a *non-trivial* LC of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ that equals $\vec{0}$. This means that

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p = \vec{0} \quad \textbf{and some scalar } \boxed{s_j \neq 0}.$$

The span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$

$$\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} = \left\{\vec{w} \mid \vec{w} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p\,, s_j \text{ scalars}\right\}$$

The *span* of a single vector is a line. Except when this is not true. $\ddot\smile$

The *span* of a two vectors is:

- a line if the two vectors are parallel
- a plane if the two vectors are not parallel
- except when this is not true $\ddot\smile$

What "is" the span of 7 vectors?

It depends. . . .

Linear Dependence

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are *linearly dependent* if there is a *non-trivial* LC of them that equals $\vec{0}$: that is, if there are scalars $s_1, s_2, \ldots, s_p$ so that

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p = \vec{0},$$

and (at least) one of the scalars is *non-zero*.

Linearly dependent vectors carry redundant information.
Vectors that are *not* LD are said to be *linearly independent*. Linearly independent vectors carry NO redundant information.

Example—three vectors in $\mathbb{R}^3$

Are the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix}$ LD or LI? Notice that $\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}3\\1\\4\end{bmatrix}$.

Thus $\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}3\\1\\4\end{bmatrix} = \vec{0}$. So the three vectors are LD.

This means that

$$\mathcal{S}pan\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix}\right\} = \mathcal{S}pan\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$$

which is a plane in $\mathbb{R}^3$.

Example—three vectors in $\mathbb{R}^3$

Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\7\end{bmatrix}$ LD or LI? Well, $2\begin{bmatrix}1\\1\\1\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\7\end{bmatrix}$, so $2\begin{bmatrix}1\\1\\1\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}2\\2\\7\end{bmatrix} = \vec{0}$. Therefore, the three vectors are LD.

This means that

$$\mathcal{S}pan\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\7\end{bmatrix}\right\} = \mathcal{S}pan\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

which is a plane in $\mathbb{R}^3$.

Example—three vectors in $\mathbb{R}^3$

Are the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ LD or LI? Since $-\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$, we get $-\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\2\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \vec{0}$. So the three vectors are LD.

This means that

$$\mathcal{S}pan\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} = \mathcal{S}pan\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$$

which is a plane in $\mathbb{R}^3$.

Example—three vectors in $\mathbb{R}^4$

Are the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 6 \\ 1 \end{bmatrix}$ LD or LI?

Easy to see that $\vec{v}_3 = 3\vec{v}_1 - \vec{v}_2$, so $3\vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \vec{0}$ and therefore $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are LD. Also, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2\}$ which is a 2-plane in $\mathbb{R}^4$.

But what if we had five (or fifteen) vectors?
Let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ be the matrix with columns given by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
Recall that $A\vec{x}$ is the LC of the columns of A given by

$$A\vec{x} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3.$$

See that there is a *non-trivial* LC $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ if and only if there is a non-trivial (i.e., non-zero) solution to $A\vec{x} = \vec{0}$.

Example—three vectors in $\mathbb{R}^4$ (continued)

We row reduce A to get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 6 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[\xrightarrow{R_4 - R_1}]{R_3 - 2*R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[\xrightarrow{R_4 - R_2}]{R_3 + 2*R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last column has no row leader, we know that x_3 is a free variable. Thus $x_3 = t$ (with t any scalar), $x_2 = t$ and $x_1 = -3t$.

There are infinitely many solutions to $A\vec{x} = \vec{0}$, and all but one of these is non-trivial. Thus, there are infinitely many non-trival LCs $s_1\vec{v}_1 + s_2\vec{v}_2 + s_3\vec{v}_3 = \vec{0}$.

Linearly Dependent Vectors

Suppose the vectors $\vec{v}, \vec{w}$ are LD. This means there are scalars s, t such that $s\vec{v} + t\vec{w} = \vec{0}$ and s, t are NOT both zero. Assume $s \neq 0$.
Then we can write $\vec{v} = -(t/s)\vec{w}$. Thus $\vec{v}$ is a scalar multiple of $\vec{w}$;
i.e., $\vec{v}$ is a LC of $\vec{w}$.
Suppose $\vec{v}_1$ is a LC of $\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_p$. This means there are scalars $s_2, s_3, \ldots, s_p$ so that

$$\vec{v}_1 = s_2\vec{v}_2 + s_3\vec{v}_3 + \cdots + s_p\vec{v}_p.$$

Then

$$\vec{v}_1 - \left(s_2\vec{v}_2 + s_3\vec{v}_3 + \cdots + s_p\vec{v}_p\right) = \vec{0}.$$

so we see that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_p$ are LD.

If some $\vec{v}_j$ is an LC of the other vectors, then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_p$ are LD.
Converse true too!

A useful Fact—"too many" vectors are always LD

If $p > n$, then any set of p vectors in $\mathbb{R}^n$ is LD.

Any 5 vectors in $\mathbb{R}^4$ are LD. Any 11 vectors in $\mathbb{R}^7$ are LD.

Why?

Just look at REF of appropriate coefficient matrix! $\ddot\smile$

Parallel Vectors

Two vectors are *parallel* if and only if one is a scalar multiple of the other.

Any two *non-zero* vectors are LI if and only if they are *not* parallel. Why? (a good quiz question!)

Suppose we have three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ (say, in $\mathbb{R}^5$) and $\vec{v}_1 \nparallel \vec{v}_2 \nparallel \vec{v}_3 \nparallel \vec{v}_1$.

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ LD or LI?

Hint: it is not difficult to produce a simple example!
(a good exam question $\ddot\smile$)