Example Illustrating the Theorem Homogeneous Solutions Sets are Parallel to Solution Sets

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Theorem 6—top of page 47

Recall that the solution set for $A\vec{x} = \vec{b}$ is the *translation* by $\vec{p}$ of the solution set for $A\vec{x} = \vec{0}$.

Here $\vec{p}$ is any solution *particular* to $A\vec{x} = \vec{b}$.

This means that $\vec{x}$ is a solution to the equation $A\vec{x} = \vec{b}$, if and only if $\vec{x} = \vec{p} + \vec{z}$ for some solution $\vec{z}$ to the homogeneous equation $A\vec{x} = \vec{0}$.

Here we work thru a simple example that illustrates this phenomenon.

Two Planes in $\mathbb{R}^3$

Let's find the solution set for the SLE

$$\begin{cases} x + y - z = 1 \\ \quad 2x + y = 2 \end{cases}.$$

Of course, geometrically, we are looking at the intersection of two planes in $\mathbb{R}^3$, which is a line.
The coefficient and augmented matrices are

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & 0 & 2 \end{array}\right].$$

Let's perform elementary row ops on the augmented matrix.

Example—some elem row ops & the solution set

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & 0 & 2 \end{array}\right] \xrightarrow{R_2 - 2*R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 2 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 + R_2} \\ \xrightarrow{-1*R_2} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

So, $z = t$ is a free variable, and $y = 2t$ and $x = 1 - t$. Thus the solution set is (algebraically) described by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\vec{m} \quad \text{where} \quad \vec{m} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and } t \text{ any scalar.}$$

Geometrically, this is a line in $\mathbb{R}^3$ which is parallel to the line $\mathbb{L} = \mathcal{S}pan\{\vec{m}\}$. Note that $\mathbb{L}$ is the solution set to the homogeneous SLE

$$\begin{cases} x + y - z = 0 \\ 2x + y = 0 \end{cases}.$$