Solutions Sets
Homogeneous Equations
Span

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Solution Sets

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We wish to better understand the structure of the solution set for an SLE

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Also, if there are infinitely many solutions, how do any two of them compare to each other?

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How does the solution set to the above LE change when we change the rhs constant b? What if we make $b = 0$?

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The solution set for $A\vec{x} = \vec{0}$ is **always** just the **span** of some vectors; always!