Solutions Sets
Homogeneous Equations
Span

Linear Algebra
MATH 2076

Solution Sets

We wish to better understand the structure of the solution set for an SLE

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}.$$

For example, if the matrix equation

$$A\vec{x} = \vec{b}$$

has a solution, and we change the rhs, do we still get a solution?
(The *homogeneous equation* $\boxed{A\vec{x} = \vec{0}}$ *always* has a solution, right?)

If there are infinitely many solutions, how do any two of them compare to each other?

Solution Sets

We also want to understand Theorem 6 on the top of page 47. We'll start by looking at SLEs with 1 equation; so a single linear equation.

Recall that, in general, the *solution set* to a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a *hyperplane* (aka, an $(n-1)$-plane) in $\mathbb{R}^n$.

(We need some $a_j \neq 0$; then we can solve for x_j in terms of the remaining variables; these remaining $n-1$ variables are the *free variables*.)

How does the solution set to the above LE change when we change the rhs constant b? What if we make $b = 0$?

Lines in $\mathbb{R}^2$

What is the solution set for $x + 2y = 3$?

How does it compare to the solution set for $x + 2y = 0$?

The solution set for $x + 2y = 3$ consists of all vectors

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{where } t \text{ can be any scalar.}$$

This is a line in $\mathbb{R}^2$ thru $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the direction $\vec{m} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Thus the solution set for $x + 2y = 3$ is parallel to $\mathbb{L}$ (the solution set for $x + 2y = 0$); it is a *translation* of the homogeneous solution set $\mathbb{L}$.

Planes in $\mathbb{R}^3$

What is the solution set for $x + 2y + 3z = 4$?

How does it compare to the solution set for $x + 2y + 3z = 0$?

The solution set for $x + 2y + 3z = 4$ consists of all vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad \text{where } s \text{ and } t \text{ can be any scalars.}$$

This is a plane in $\mathbb{R}^3$ thru $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ and parallel to the plane $\mathbb{W} = \mathcal{S}pan\{\vec{v}, \vec{w}\}$.

Thus the solution set for $x + 2y + 3z = 4$ is parallel to $\mathbb{W}$ (the solution set for $x + 2y + 3z = 0$); it is a *translation* of the homogeneous solution set $\mathbb{W}$.

Theorem 6—top of page 47

Let $\vec{p}$ be a solution to $A\vec{x} = \vec{b}$; we call $\vec{p}$ a *particular* solution.

If $\vec{z}$ is a solution to the homogeneous equation $A\vec{x} = \vec{0}$, then
$\vec{x} = \vec{p} + \vec{z}$ is a solution to $A\vec{x} = \vec{b}$.

If $\vec{x}$ is a solution to the equation $A\vec{x} = \vec{b}$, then
$\vec{z} = \vec{x} - \vec{p}$ is a solution to the homogeneous equation $A\vec{x} = \vec{0}$.

Thus the solution set for $A\vec{x} = \vec{b}$ is the *translation* by $\vec{p}$
of the solution set for $A\vec{x} = \vec{0}$. These two solution sets are *parallel*.

To better understand the solution set for $A\vec{x} = \vec{b}$, we need
to better understand the solution set for $A\vec{x} = \vec{0}$.

The solution set for $A\vec{x} = \vec{0}$ is **always** just the **span** of some vectors; always!