Vector Equations
Matrix Equations
Linear Combinations

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Three Views of Same Idea

Let $\vec{a}_j$ be the j^{th} column of the coefficient matrix A for some SLE

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}.$$

The above SLE has exactly the same solutions as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

which in turn has exactly the same solutions as the matrix equation

$$A\vec{x} = \vec{b}.$$

REF Solution Information

Look at a REF for the *augmented* matrix associated with the SLE.
If the last column has a row leader, there are NO solutions;
assume otherwise.

Identify the columns that do *not* have row leaders; the corresponding variables are *free*.

If there are no free variables, get a unique solution; if there are free variables, get infinitely many solutions and need to describe <u>all</u> of them.

Each *basic* (aka,non-free) variable can be expressed in terms of the free variables. We can do this by back substitution, or by continuing to row reduce to get reduced REF.

Linear Combinations

Suppose $s_1, s_2, \ldots, s_p$ are scalars and $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$ are vectors (all in the same space $\mathbb{R}^n$). We call

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p$$

a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$. For example, we always have the *trivial* linear combination

$$0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p = \vec{0}.$$

We'll be most interested in the set of *all* LCs of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$. We call this the *span* of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$, and write is as

$$\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}.$$

Solutions

Remember that a vector equation such as

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

has a solution if and only if the rhs $\vec{b}$ is a LC of $\vec{a}_1, \ldots, \vec{a}_n$.

This means that there is a solution if and only if $\vec{b}$ belongs to

$$\mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\}\,.$$

This does not tell us how to find the solution.

Theorem 4—bottom of page 37 in Section 1.4

Look at this!