Vector Equations
Matrix Equations
Linear Combinations

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Three Views of Same Idea

Let $\vec{a}_j$ be the j^{th} column of the coefficient matrix A for some SLE

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} .$$

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So

$$\vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad \text{and} \quad A = \left[\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_n\right] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

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which in turn has exactly the same solutions as the matrix equation

$$A\vec{x} = \vec{b}.$$

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Linear Combinations

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$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p$$

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We'll be most interested in the set of *all* LCs of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$. We call this the *span* of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$, and write is as

$$\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}.$$

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This does not tell us how to find the solution.

Theorem 4—bottom of page 37 in Section 1.4

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Look at this!