Vectors, Equations, Linear Combinations

Linear Algebra
MATH 2076

Matrices

Recall that a *matrix* is a rectangular array of numbers.

We call

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

an $m \times n$ matrix; it has m *rows* and n *columns*.

We write $\mathbb{R}^{m \times n}$ for the space of all $m \times n$ matrices.

Vectors

Matrices that have just a single row, or a single column, are often called *row vectors*, or *column vectors*. For us, a *vector* is always a column vector, that is, a matrix with exactly one column. So, a vector is an $n \times 1$ matrix, where n is the number of rows (or entries, or coordinates).

It is useful to adopt a notation that helps distinguish between numbers (aka scalars), matrices, and vectors; I'll use arrows. Thus a vector of "size" n (i.e., with n coordinates) is written as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where } x_1, x_2, \ldots, x_n \text{ are the coordinates of } \vec{x}.$$

The space of all such vectors, of "size" n, is $\mathbb{R}^{n \times 1}$. We usually just write $\mathbb{R}^n$, but this is not exactly correct, is it?

Vector Arithmetic

We can add two vectors, provided they have the same number of coordinates. If $\vec{x}$ and $\vec{y}$ are both in $\mathbb{R}^n$, say with coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, then

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + yn \end{bmatrix}.$$

We can also multiply vectors by scalars. If s is any scalar, then

$$s\vec{x} = s \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s\,x_1 \\ s\,x_2 \\ \vdots \\ s\,x_n \end{bmatrix}.$$

Linear Combinations

Suppose $s_1, s_2, \ldots, s_p$ are scalars and $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$ are vectors (all in the same space $\mathbb{R}^n$). We call

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p$$

a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$.
For example, we always have the *trivial* linear combination

$$0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p = \vec{0}.$$

Consider the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Notice that

$$x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus every vector in $\mathbb{R}^3$ can be expressed as a LC of $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

Vector Equations

Let $\vec{a}_1, \ldots, \vec{a}_n$ and $\vec{b}$ be given vectors (all in the same space $\mathbb{R}^m$). Then

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

is called a *vector equation*; here $x_1, x_2, \ldots, x_n$ are the variables.
For example, the vector equation

$$x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has solution $x_1 = b_1, x_2 = b_2, x_3 = b_3$, or more simply, $\vec{x} = \vec{b}$.

Notice that the above vector equation (at top of page) has a solution if and only if the rhs $\vec{b}$ is a LC of $\vec{a}_1, \ldots, \vec{a}_n$. Right?
If A is the coefficient matrix for some SLE, and $\vec{a}_j$ is the j^{th} column of A, then

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

has exactly the same solutions as the original SLE.
Thus, an SLE has a solution if and only if ....

Matrix Multiplication—preliminary product

Let $\vec{x}$ be a vector in $\mathbb{R}^n$ and A a matrix in $\mathbb{R}^{m\times n}$. So

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The product $A\vec{x}$ is defined to be the LC

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

where $\vec{a}_j$ is the j^{th} column of A.

If $\vec{b}$ is a vector in $\mathbb{R}^m$, we call $\boxed{A\vec{x} = \vec{b}}$ a *matrix equation*; here $\vec{x}$ is the variable.

Three Views of Same Idea

Let $\vec{a}_j$ be the j^{th} column of the coefficient matrix A for some SLE

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}.$$

The above SLE has exactly the same solutions as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

which in turn has exactly the same solutions as the matrix equation

$$A\vec{x} = \vec{b}.$$