Vectors, Equations, Linear Combinations

Linear Algebra
MATH 2076

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We write $\mathbb{R}^{m \times n}$ for the space of all $m \times n$ matrices.

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$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + yn \end{bmatrix}.$$

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Linear Combinations

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Thus every vector in $\mathbb{R}^3$ can be expressed as a LC of $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

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If A is the coefficient matrix for some SLE, and $\vec{a}_j$ is the j^{th} column of A, then

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Thus, an SLE has a solution if and only if . . . .

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The product $A\vec{x}$ is defined to be the LC

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If $\vec{b}$ is a vector in $\mathbb{R}^m$, we call $\boxed{A\vec{x} = \vec{b}}$ a *matrix equation*; here $\vec{x}$ is the variable.

Three Views of Same Idea

Let $\vec{a}_j$ be the j^{th} column of the coefficient matrix A for some SLE

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The above SLE has exactly the same solutions as the vector equation

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which in turn has exactly the same solutions as the matrix equation

$$A\vec{x} = \vec{b}.$$