Row Echelon Form

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Solving Systems of Linear Equations

To solve a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

we use elementary row operations to find a REF for the associated augmented matrix.

All questions about solns to SLEs can be answered by looking at a REF of the SLE's augmented matrix. Recall that *row echelon form* means:

- all zero rows at the bottom,
- *row leaders* move to right as go down.

A *row leader* in a non-zero row is the first non-zero entry.
Above conditions mean every entry below a row leader must be zero.

Elementary Row Operations

The following are allowable elementary row operations. (These have the property that they do not alter the solution set.)

- Add a multiple of one row to another.
- Multiply one row by a *non-zero* constant.
- Interchange two rows.

By repeatedly applying these ops, one at a time, we can convert any matrix into a REF.

We can even convert to *reduced* REF; here every entry both below and *above* a row leader must be zero.

Example

Consider the SLE

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 7y - 8z = 0 \end{cases}.$$

The coefficient and augmented matrices are

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 7 & -8 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 7 & -8 & 0 \end{array}\right].$$

Let's perform elementary row ops on the augmented matrix.

Example—doing elementary row ops to rows 2 and 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 7 & -8 & 0 \end{array}\right] \xrightarrow{R_2 - 2*R_1} \xrightarrow{R_3 - 3*R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 4 & -14 & -27 \end{array}\right]$$

$$\xrightarrow{R_3 - 2*R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

The last row corresponds to the equation $0 \cdot x + 0 \cdot y + 0 \cdot z = 7$, which has no solutions. Thus the original SLE has no solutions.

Example

All questions about solns to SLEs can be answered by looking at a REF of the SLEs augmented matrix.

Suppose the augmented matrix for some SLE has the following REF.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

What can we say?

REF Solution Information

Let E be a REF for the *augmented* matrix associated with some SLE. If the last column of E has a row leader, there are NO solutions; assume otherwise.

Identify the columns of E that do *not* have row leaders; the corresponding variables are *free*.

If there are no free variables, get a unique solution; if there are free variables, get infinitely many solutions and need to describe all of them.

Each *basic* (aka,non-free) variable can be expressed in terms of the free variables. We can do this by back substitution, or by continuing to row reduce to get the reduced REF.

The *pivot columns* are the columns in the *original* matrix that correspond to the basic variables.