Row Echelon Form

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Solving Systems of Linear Equations

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Above conditions mean every entry below a row leader must be zero.

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We can even convert to *reduced* REF; here every entry both below and *above* a row leader must be zero.

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Consider the SLE

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 7y - 8z = 0 \end{cases}.$$

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The coefficient and augmented matrices are

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 7 & -8 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 7 & -8 & 0 \end{array}\right].$$

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Let's perform elementary row ops on the augmented matrix.

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The *pivot columns* are the columns in the *original* matrix that correspond to the basic variables.