Solving Systems of Linear Equations

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

SLE Solution Sets

SLE Solution Set Trichotomy

Every system of linear equations has exactly

- 0 solutions (that is, no solution), or,
- 1 solution (so, a *unique* solution), or,
- infinitely many solutions.

But, how do we find the solution set?

Solving Systems of Linear Equations

To solve a system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

we use elementary operations to convert it into an *equivalent* upper triangular system; *equivalent* SLEs have exactly the same solution set.

An upper triangular system is easy to solve by using *back substitution*.

So, what are the allowable operations? These must have the property that they do not alter the solution set.

Elementary Operations

None of the following operations changes the solution set.

- Add a multiple of one equation to another.
- Multiply one equation by a *non-zero* constant.
- Interchange two equations.

By repeatedly applying these ops, one at a time, we can convert any SLE into an upper triangular SLE.

Notice that we don't really need the variables, right? That is, we only need to keep track of how the coeffs and rhs constants change.

The basic info for any SLE is given by its coeffs and rhs constants. This info can be recorded compactly in a *matrix*.

Coefficient Matrix and Augmented Matrix

A *matrix* is a rectangular array of numbers.
The *coefficient matrix* for the SLE

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

is the matrix that has a_{ij} in its i^{th} row and j^{th} column.

Example

Consider the SLE

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

The coefficient and augmented matrices are

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right].$$

Let's perform elementary row ops on the augmented matrix. Remember, these do not change the solution set!

Example—doing elementary row ops to row 3

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{R_3+4*R_1} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{2*R_3} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -6 & 26 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+3*R_2} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

So, $x_3 = 3$; can back sub now.

$$\xrightarrow{\frac{1}{2}*R_3} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example—doing more elementary row ops

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\frac{1}{2}*R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_2+4*R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1+2*R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 32 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So, $x_1 = 29, x_2 = 16, x_3 = 3.$

Example—the solution

Thus the SLE

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

has the unique solution $\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$.

Notice the form of the final augmented matrix! It was $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$.

Row Echelon Form

A matrix is in *row echelon form* provided

- All its zero rows are at the bottom.
- Its *row leaders* move to right as go down.

The *row leader* in a non-zero row is the first non-zero entry.
Every entry below a row leader must be zero.

All questions about solns to SLEs can be answered by looking at a REF of the SLE's augmented matrix. For example, suppose the last column of such an REF has a row leader. Then...?

Hint: What is the corresponding equation? Isn't it something like...
$0x_1 + 0x_2 + \cdots + 0x_n = b$ for some number $b \neq 0$. So?