Solving Systems of Linear Equations

Linear Algebra
MATH 2076

SLE Solution Sets

SLE Solution Set Trichotomy

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- infinitely many solutions.

But, how do we find the solution set?

Solving Systems of Linear Equations

To solve a system of linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

we use elementary operations to convert it into an *equivalent* upper triangular system; *equivalent* SLEs have exactly the same solution set.

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we use elementary operations to convert it into an *equivalent* upper triangular system; *equivalent* SLEs have exactly the same solution set.

An upper triangular system is easy to solve by using *back substitution*.

So, what are the allowable operations? These must have the property that they do not alter the solution set.

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Notice that we don't really need the variables, right? That is, we only need to keep track of how the coeffs and rhs constants change.

The basic info for any SLE is given by its coeffs and rhs constants. This info can be recorded compactly in a *matrix*.

Coefficient Matrix and Augmented Matrix

A *matrix* is a rectangular array of numbers.

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The *coefficient matrix* for the SLE

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

is the matrix that has a_{ij} in its i^{th} row and j^{th} column.

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$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Coefficient Matrix and Augmented Matrix

A *matrix* is a rectangular array of numbers.
Including the right-hand-side constants, we get the *augmented matrix*

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Example

Consider the SLE

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

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The coefficient and augmented matrices are

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right].$$

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Let's perform elementary row ops on the augmented matrix. Remember, these do not change the solution set!

Example—doing elementary row ops to row 3

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

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So, $x_3 = 3$; can back sub now.

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So, $x_1 = 29, x_2 = 16, x_3 = 3.$

Example—the solution

Thus the SLE

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

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Notice the form of the final augmented matrix!

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Notice the form of the final augmented matrix! It was $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$.

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$0x_1 + 0x_2 + \cdots + 0x_n = b$ for some number $b \neq 0$. So?