Solving Systems of LEs

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

SLE Solution Sets

SLE Solution Trichotomy

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- 0 solutions (that is, no solution), or,
- 1 solution (so, a *unique* solution), or,
- infinitely many solutions.

Solving Systems of Linear Equations

To solve a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

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So, what are allowable operations? These must have the property that they do not alter the solution set.

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The basic info for any SLE is given by its coeffs and rhs constants. This info can be recorded compactly in a *matrix*.

Coefficient Matrix and Augmented Matrix

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The *coefficient matrix* for the SLE

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is the matrix that has a_{ij} in its i^{th} row and j^{th} column.

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$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Coefficient Matrix and Augmented Matrix

A *matrix* is a rectangular array of numbers.

Including the right-hand-side constants, we get the *augmented matrix*

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Example

Consider the SLE

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}.$$

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The coefficient and augmented matrices are

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right].$$

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Let's perform elementary row ops on the augmented matrix.

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So, $x_3 = 3$; can back sub now.

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So, $x_1 = 29, x_2 = 16, x_3 = 3.$

Example—the solution

Thus the SLE

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

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Notice the form of the final augmented matrix!

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$0x_1 + 0x_2 + \cdots + 0x_n = b$ for some number $b \neq 0$. So?