Systems of Linear Equations

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Linear Equations and their Solutions

A *linear equation* in "unknowns" (the *variables*) $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Here $a_1, a_2, \ldots, a_n$ are the *coefficients* and b is the *right-hand-side*; these are numbers (aka, *constants* or *scalars*) that are usually—but not always—known in advance.

A *solution* to the above linear equation is a list $(s_1, s_2, \ldots, s_n)$ of numbers such that setting $x_1 = s_1, \ldots, x_n = s_n$ makes the equation a true statement. The *solution set* consists of **all** of the solutions.

Systems of Linear Equations and their Solutions

A *system* of linear equation in "unknowns" (the *variables*) $x_1, x_2, \ldots, x_n$ has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Here $a_{11}, a_{12}, \ldots, a_{mn}$ are the *coefficients* and $b_1, \ldots, b_m$ are the *right-hand-side constants*; these are numbers (aka, *constants* or *scalars*) that are usually—but not always—known in advance.

A *solution* to the above system is a list $(s_1, s_2, \ldots, s_n)$ of numbers such that setting $x_1 = s_1, \ldots, x_n = s_n$ makes **all** of the equations true statements. The *solution set* consists of **all** solutions.

SLE Solution Sets

SLE Solution Set Trichotomy

Every system of linear equations has exactly

- 0 solutions (that is, no solution), or,
- 1 solution (so, a *unique* solution), or,
- infinitely many solutions.

In the third case, when there are infinitely many solutions, we must—somehow—describe *all* of solutions.

There are two totally different ways to go about describing the solution set for an SLE. An *algebraic description* for the solution set provides a formula, or formulas, that gives every single solution.

A *geometric description* for the solution set provides a pictorial visualization that represents every single solution.

Euclidean Space

The set $\mathbb{R}^2 = \{(x, y) \mid x, y \text{ any numbers}\}$ is called 2-*dimensional Euclidean space*; the xy-plane is a *geometric visualization* for $\mathbb{R}^2$ and allows us to draw pictures that describe certain subsets of $\mathbb{R}^2$.

The set $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \text{ any numbers}\}$ is called 3-*dimensional Euclidean space*; xyz-space is a *geometric visualization* for $\mathbb{R}^3$ and allows us to draw pictures that describe certain subsets of $\mathbb{R}^2$.

The set $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \text{ any numbers}\}$ is called *n-dimensional Euclidean space*. 'Seeing' a *geometric visualization* for $\mathbb{R}^n$ is difficult and drawing pictures that describe subsets of $\mathbb{R}^n$ can be a daunting task; but, we can do simple things!

Notice that the solution set to an SLE with n variables is a subset of $\mathbb{R}^n$.

Solution Set for One Linear Equation

Again, a *solution* to the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a list $(s_1, s_2, \ldots, s_n)$ of numbers such that setting $x_1 = s_1, \ldots, x_n = s_n$ makes the equation true. We want to "find" the *solution set* which consists of **all** of the solutions.
It is easy to find a few solutions; just set all but one of the variables equal to 0 or 1 or 2 or . . . .

In general, if, say $a_j \neq 0$, we can solve for x_j in terms of the remaining variables; these remaining $n - 1$ variables are called *free variables* because they can be anything (i.e., any numbers). In this setting, our solution set has $n - 1$ degrees of freedom, and we call it a *hyperplane* in $\mathbb{R}^n$.

Hyperplanes in $\mathbb{R}^n$

A *hyperplane* in $\mathbb{R}^n$ is the *solution set* to a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where at least one coefficient is non-zero.

When $n = 2$ we get $ax + by = c$ which gives a line in $\mathbb{R}^2$.

When $n = 3$ we get $ax + by + cz = d$ which gives a plane in $\mathbb{R}^3$.

When $n = 4$ we get $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$ which gives a hyperplane in $\mathbb{R}^4$.

Solution Set for System of Linear Equations

Again, a *solution* to the SLE

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

is a list $(s_1, s_2, \ldots, s_n)$ of numbers such that setting $x_1 = s_1, \ldots, x_n = s_n$ makes all of the equations true. We want to "find" the *solution set* which is **all** of the solutions.
Now it is not easy to find even one solution.

In fact, the solution set is the intersection of the m hyperplanes given by the m equations.

So, how can we: Visualize this intersection? Describe all solutions?