

Systems of Linear Equations

Linear Algebra
MATH 2076



Linear Equations and their Solutions

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A *solution* to the above linear equation is a list (s_1, s_2, \dots, s_n) of numbers such that setting $x_1 = s_1, \dots, x_n = s_n$ makes the equation a true statement. The *solution set* consists of **all** of the solutions.

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There are two totally different ways to go about describing the solution set for an SLE. An *algebraic description* for the solution set provides a formula, or formulas, that gives every single solution.

A *geometric description* for the solution set provides a pictorial visualization that represents every single solution.

Euclidean Space

The set $\mathbb{R}^2 = \{(x, y) \mid x, y \text{ any numbers}\}$ is called *2-dimensional Euclidean space*; the xy -plane is a *geometric visualization* for \mathbb{R}^2 and allows us to draw pictures that describe certain subsets of \mathbb{R}^2 .

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The set $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \text{ any numbers}\}$ is called *3-dimensional Euclidean space*; xyz -space is a *geometric visualization* for \mathbb{R}^3 and allows us to draw pictures that describe certain subsets of \mathbb{R}^3 .

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Notice that the solution set to an SLE with n variables is a subset of \mathbb{R}^n .

Solution Set for One Linear Equation

Again, a *solution* to the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a list (s_1, s_2, \dots, s_n) of numbers such that setting $x_1 = s_1, \dots, x_n = s_n$ makes the equation true. We want to “find” the *solution set* which consists of **all** of the solutions.

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In general, if, say $a_j \neq 0$, we can solve for x_j in terms of the remaining variables; these remaining $n - 1$ variables are called *free variables* because they can be anything (i.e., any numbers). In this setting, our solution set has $n - 1$ degrees of freedom, and we call it a *hyperplane* in \mathbb{R}^n .

Hyperplanes in \mathbb{R}^n

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When $n = 4$ we get $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$ which gives a hyperplane in \mathbb{R}^4 .

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