Determinants—an Introduction

Applied Linear Algebra
MATH 5112/6012

What is a Determinant?

For each square matrix A, get associated number $\det(A)$ with properties:

- A is invertible if and only if $\det(A) \neq 0$
- $\det(A) = \pm\text{vol}(\Pi)$ where Π is image of unit cube under $\vec{x} \mapsto A\vec{x}$

Thus have function $\mathbb{R}^{n\times n} \xrightarrow{\det} \mathbb{R}$ where $A \mapsto \det(A)$.

Calculating $\det(A)$ is a **terrible** way to determine if A is invertible!

What is a Determinant?

The determinant function $\mathbb{R}^{n\times n} \xrightarrow{\text{det}} \mathbb{R}$ is defined recursively.

To find the determinant of an $n \times n$ matrix, we need to know how to find the determinant of an $(n-1) \times (n-1)$ matrix.

The determinant of a 2×2 matrix is easy to calculate:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

For example,

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2.$$

It's convenient to write $|A| = \det(A)$. So, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2.$

The Determinant of a 3×3 Matrix

The determinant of a 3×3 matrix is given by

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - ge)$$

This is called *cofactor expansion across the first row*. For example,

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9 = 0.$$

The Determinant of a 4×4 Matrix

The determinant of a 4×4 matrix is given by

$$\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix} =$$

$$= a \begin{vmatrix} f & g & h \\ j & k & l \\ n & p & q \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & p & q \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & q \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & p \end{vmatrix}.$$

This is called *cofactor expansion across the first row*.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of certain $(n-1) \times (n-1)$ matrices called the *minors* of A.

The (i, j)-*minor of* A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting both the i^{th} row and j^{th} column of A:

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

E.g., the $(2, 3)$ minor of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $\begin{bmatrix} a & b \\ g & h \end{bmatrix}$.

The Determinant of an $n \times n$ Matrix

The determinant of an $n \times n$ matrix A is given in terms of determinants of its minors of M_{ij}. We have

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}) \\ &= a_{11}\left|M_{11}\right| - a_{12}\left|M_{12}\right| + \cdots + (-1)^{1+n} a_{1n}\left|M_{1n}\right|.\end{aligned}$$

This is called *cofactor expansion across the first row*. In fact, we can calculate $\det(A)$ by expanding across any row

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion across the } i^{\text{th}} \text{ row)}$$

or by expanding down any column

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{(cofactor expansion down the } j^{\text{th}} \text{ column)}.$$

Example

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Determinant of Upper Triangular Matrix

Recall that a square matrix is *upper triangular* if all of its entries below the main diagonal are zero. Easy to see that for an upper triangular $n \times n$ matrix $A = [a_{ij}]$ we have

$$\det(A) = a_{11}a_{22}\ldots a_{nn}.$$

Recall that by repeatedly applying elem row ops, one at a time, we can convert any square matrix into an upper triangular matrix.

The following are allowable elementary row operations.

- Add a multiple of one row to another.
- Multiply one row by a *non-zero* constant.
- Interchange two rows.

How do these elem row ops change the determinant?

Determinants and Elementary Row operations

The following are allowable elementary row operations.

1. Add a multiple of one row to another.
2. Multiply one row by a *non-zero* constant k.
3. Interchange two rows.

How do these elem row ops change the determinant?

Let A be a square matrix; so $\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(M_{ij})$.
Suppose we perform an elem row op on A to get B. Then:

- $\det(B) = \det(A)$ for a type (1) elem row op ($\ddot{\smile}$)
- $\det(B) = k\det(A)$ for a type (2) elem row op
- $\det(B) = -\det(A)$ for a type (3) elem row op

Examples

Find the determinants of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 3 & 4 & 1 & 0 & -1 \\ 6 & 4 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 & 4 \\ 3 & 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Answers: $\det(A) = 18$ and $\det(B) = -100$.

Properties of Determinants

Let A and B be square matrices of the same size. Then:

- $\det(AB) = \det(A)\det(B)$
- $\det(kA) = k^n \det(A)$ (if A is $n \times n$)
- $\det(A^T) = \det(A)$
- If A is invertible, then $\det(A^{-1}) = \big(\det(A)\big)^{-1}$