Geometric Transformations

Applied Linear Algebra
MATH 5112/6012

UNIVERSITY OF Cincinnati

Geometric Transformations

These are transformations $\mathbb{R}^2 \to \mathbb{R}^2$ that include

- translations
- dilations
- rotations
- reflections
- projections
- shearing

By using compositions of these, we can create all sorts of transformations.

Many of the above can also be defined as maps $\mathbb{R}^n \to \mathbb{R}^n$.

Translations and Dilations in $\mathbb{R}^2$

A *translation of* $\mathbb{R}^2$ *by* $\vec{a}$ is the map $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \vec{a};$$

here $\vec{a}$ is some fixed vector in $\mathbb{R}^2$.

A *dilation/scaling of* $\mathbb{R}^2$ *by* k is the map $\mathbb{R}^2 \xrightarrow{S} \mathbb{R}^2$ defined by

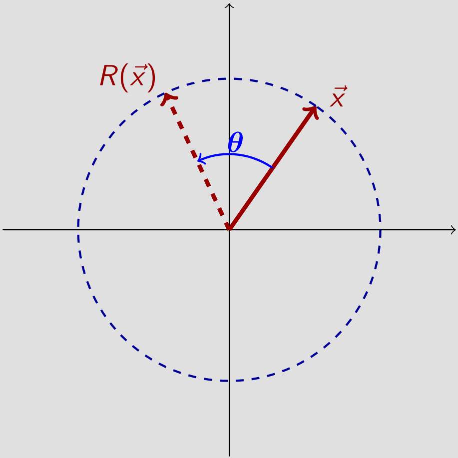
$$S(\vec{x}) = k\vec{x};$$

here $k > 0$ is some fixed positive scalar.

Both translations and dilations can be defined in $\mathbb{R}^n$; we can even use exactly the same formulas.

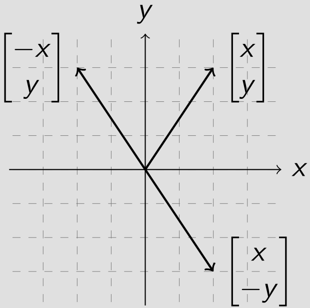
Rotations in $\mathbb{R}^2$

A *rotation of* $\mathbb{R}^2$ *by* θ is the map $\mathbb{R}^2 \xrightarrow{R} \mathbb{R}^2$ defined by letting $R(\vec{x})$ be the vector obtained by rotating $\vec{x}$ (about $\vec{0}$) by θ radians (in the clockwise direction).



Reflections in $\mathbb{R}^2$

The *reflection* $\mathbb{R}^2 \xrightarrow{R} \mathbb{R}^2$ *across* $\mathbb{L}$ is given by letting $R(\vec{x})$ be the vector obtained by reflecting $\vec{x}$ across the line $\mathbb{L}$; $\mathbb{L}$ is some fixed line in $\mathbb{R}^2$.



Reflection across the x-axis is

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Reflection across the y-axis is

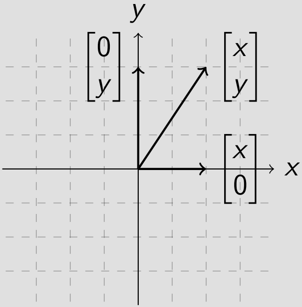
$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Reflection across the line $y = x$ is

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}.$$

Projections in $\mathbb{R}^2$

The *projection* $\mathbb{R}^2 \xrightarrow{P} \mathbb{R}^2$ *onto* $\mathbb{L}$ is given by letting $P(\vec{x})$ be the vector obtained by orthogonally projecting $\vec{x}$ onto the direction vector for the line $\mathbb{L}$; $\mathbb{L}$ is some fixed line in $\mathbb{R}^2$.



Projection onto the x-axis is

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Projection onto the y-axis is

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

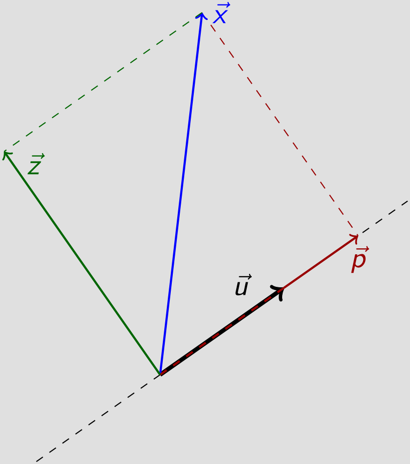
Projection onto the line $y = x$,

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{x+y}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We will discuss projections at great length in Chapter 6!

Orthogonal Projection Onto a Vector

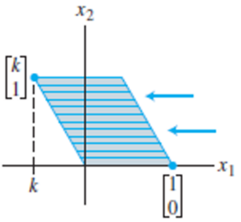
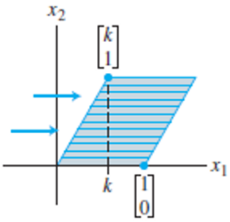
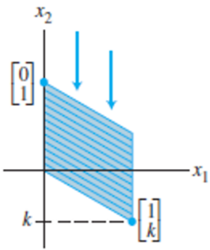
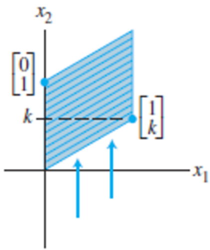
Let $\vec{u}$ be a fixed vector, and $\vec{x}$ a variable vector.



The *orthogonal projection of $\vec{x}$ onto $\vec{u}$* is the pictured vector $\vec{p}$ which is parallel to $\vec{u}$ (so, $\vec{p} = s\vec{u}$ for some scalar) and has the property that $\vec{z} = \vec{x} - \vec{p} \perp \vec{u}$.

In Chapter 6 we will see that it is easy to determine s.

TABLE 3 Shears

Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$