Geometric Transformations

Applied Linear Algebra
MATH 5112/6012

UNIVERSITY OF Cincinnati

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Many of the above can also be defined as maps $\mathbb{R}^n \to \mathbb{R}^n$.

Translations and Dilations in $\mathbb{R}^2$

A *translation of* $\mathbb{R}^2$ *by* $\vec{a}$ is the map $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ defined by

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Both translations and dilations can be defined in $\mathbb{R}^n$; we can even use exactly the same formulas.

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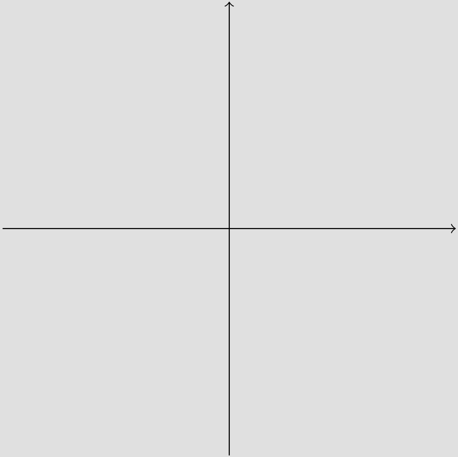

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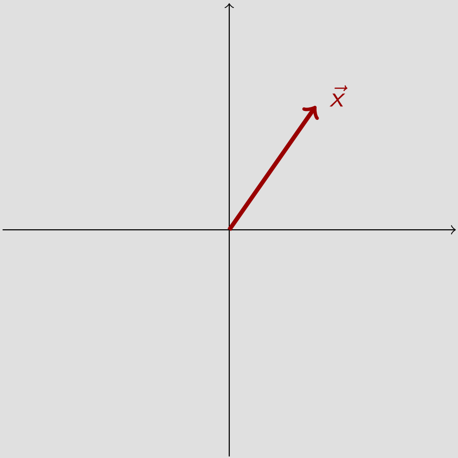

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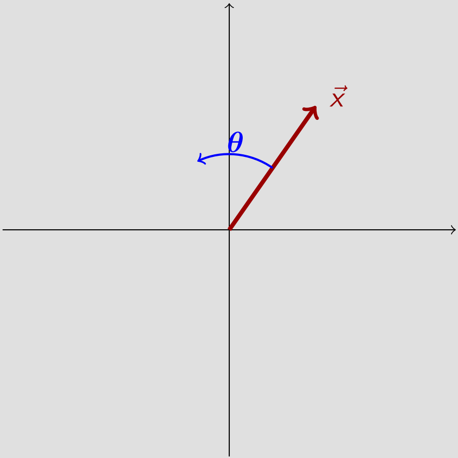

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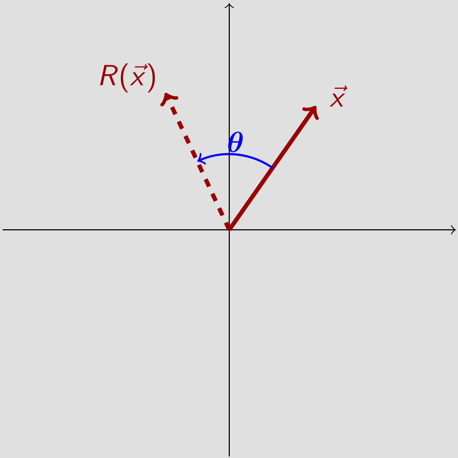

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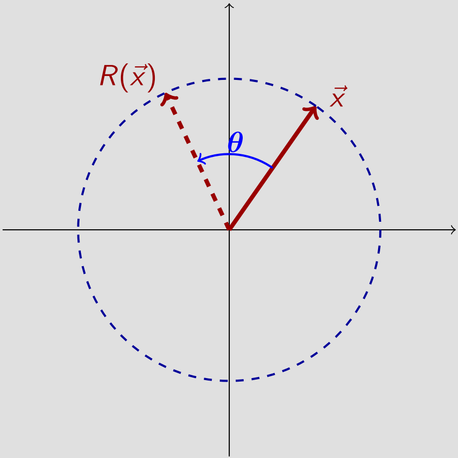

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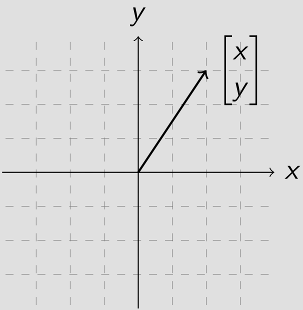

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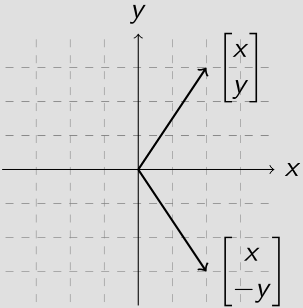

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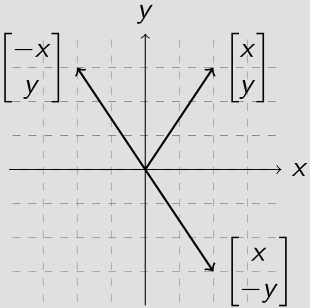

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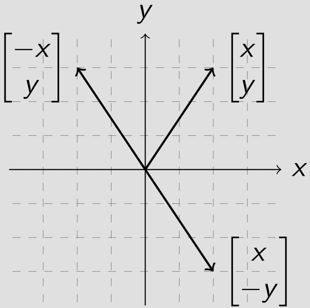

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Reflection across the line $y = x$ is

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The *projection* $\mathbb{R}^2 \xrightarrow{P} \mathbb{R}^2$ *onto* $\mathbb{L}$ is given by letting $P(\vec{x})$ be the vector obtained by orthogonally projecting $\vec{x}$ onto the direction vector for the line $\mathbb{L}$;

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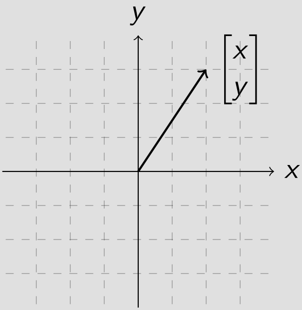

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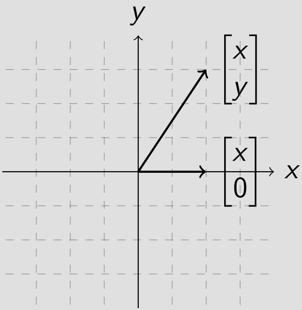

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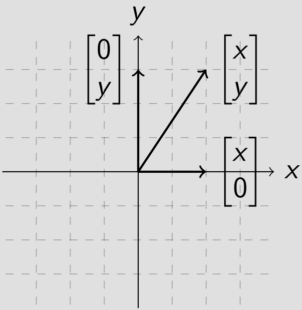

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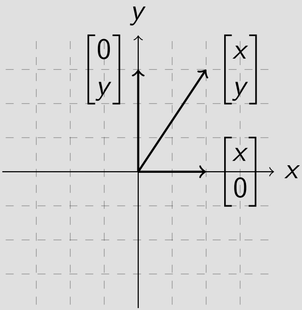

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Projection onto the line $y = x$,

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{x+y}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We will discuss projections at great length in Chapter 6!

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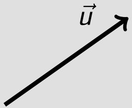

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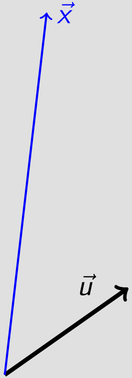

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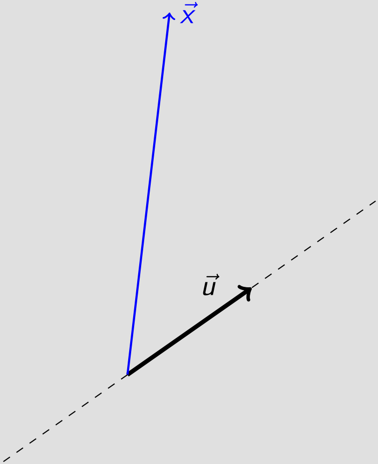

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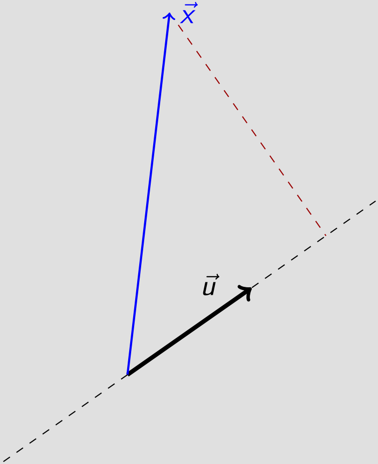

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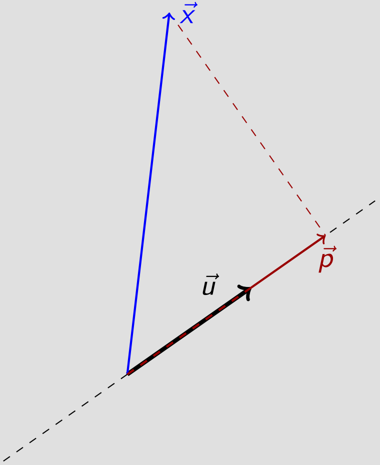

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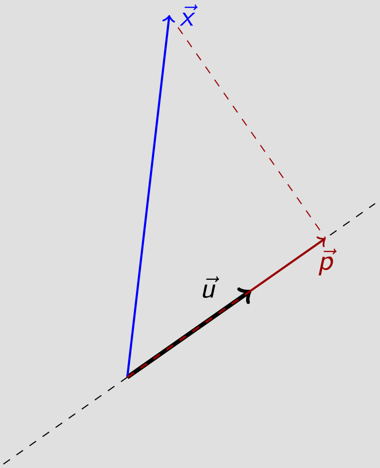

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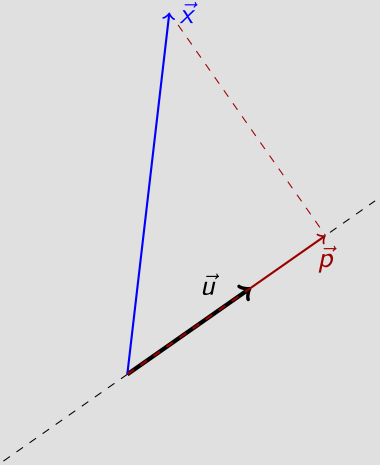

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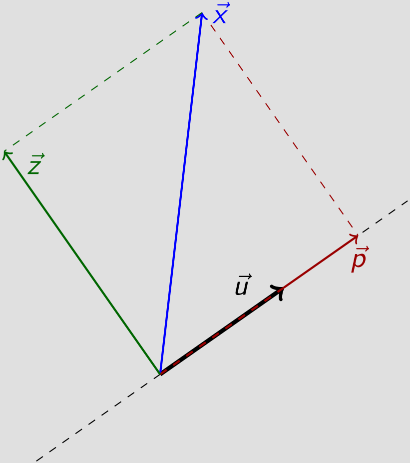

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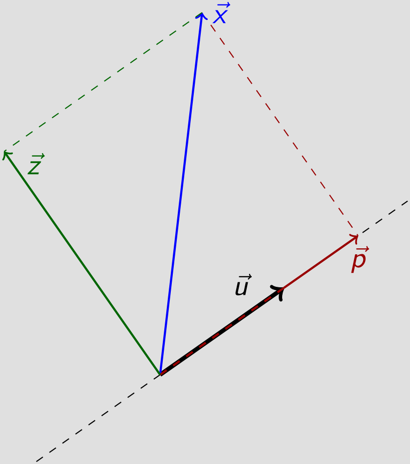

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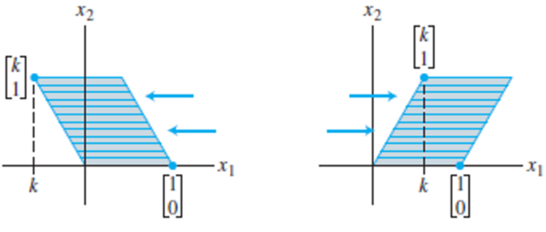
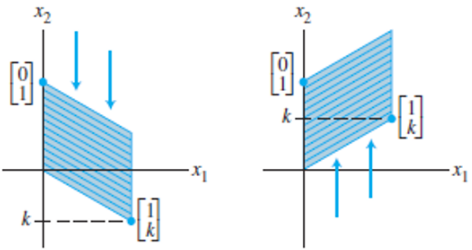
In Chapter 6 we will see that it is easy to determine s.

TABLE 3 Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$