Linear Transformations
Chapter 3, Section 6
An Example

Applied Linear Algebra
MATH 5112/6012

UNIVERSITY OF Cincinnati

Define $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$.

- Find the domain, codomain, and range of T.
- Determine whether or not $\vec{b}$ belongs to the range of T, and if so, find all vectors $\vec{x}$ such that the T image of $\vec{x}$ is $\vec{b}$, where

$$\vec{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}.$$

First, A is a 4×3 matrix, so $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^4$. That is,

- the domain of T is $\mathbb{R}^3$,
- the codomain is $\mathbb{R}^4$.

Finding the range of T requires more work.

First—the range of T

What is the range of T?

$\mathcal{R}ng(T)$ is just all possible images $T(\vec{x})$, for all possible $\vec{x}$ in $\mathbb{R}^3$.
That is, all possible $T(\vec{x}) = A\vec{x}$.

Let $\vec{a}_1, \vec{a}_2, \vec{a}_3$ be the columns of A. Then, $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$ where x_1, x_2, x_3 are the coordinates of $\vec{x}$.

So, we want all possible $T(\vec{x}) = A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$.

This is all possible LCs of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. This is $\mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

That is, $\boxed{\mathcal{R}ng(T) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}}$; $\mathcal{R}ng(T) = \mathcal{CS}(A)$.

There are two different ways to proceed.

The range of T is $\mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

Let's row reduce $A = \left[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\right]$. Not hard to see that

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, what exactly does this tell us? We see that

$$A\vec{x} = \vec{0} \iff \vec{x} = t\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{where } t \text{ is any scalar,}$$

so the solution set to $A\vec{x} = \vec{0}$ is exactly $\mathbb{L} = \mathcal{S}pan\left\{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}\right\}$;

this is a line in $\mathbb{R}^3$ thru $\vec{0}$.

Finding $\mathcal{R}ng(T) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

Thus the solution set to $A\vec{x} = \vec{0}$ is the line $\mathbb{L} = \mathcal{S}pan\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

So, for example, $A \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \vec{0}$, which says that $3\vec{a}_1 + \vec{a}_2 - \vec{a}_3 = \vec{0}$.

That is, $\vec{a}_3 = 3\vec{a}_1 + \vec{a}_2$. Therefore, $\mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2\}$.

We have found $\mathcal{R}ng(T) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2\} = \mathcal{S}pan\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 1 \\ 5 \end{bmatrix} \right\}$.

This is a 2-plane in $\mathbb{R}^4$ thru $\vec{0}$.
Note that $\vec{a}_1$ and $\vec{a}_2$ are the pivot columns of A, right?

Last part of problem

We now know that $\mathcal{R}ng(T)$ is the 2-plane $\mathcal{S}pan\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 1 \\ 5 \end{bmatrix} \right\}$.

Now we must... Determine whether or not $\vec{b}$ belongs to the range of T, and if so, find all vectors $\vec{x}$ such that the T image of $\vec{x}$ is $\vec{b}$, where

$$\vec{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}.$$

Can we find scalars s_1, s_2 so that

$$\vec{b} = s_1 \vec{a}_1 + s_2 \vec{a}_2 = s_1 \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \end{bmatrix} + s_2 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 5 \end{bmatrix} ?$$

Is $\vec{b}$ in $\mathcal{R}ng(T)$?

Actually, it is easy to find scalars s_1, s_2 so that

$$\begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix} = \vec{b} = s_1\vec{a}_1 + s_2\vec{a}_2 = s_1 \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \end{bmatrix} + s_2 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} s_1 - 2s_2 \\ 3s_1 - 4s_2 \\ s_2 \\ -3s_1 + 5s_2 \end{bmatrix}$$

We see that $s_2 = 3$, and $s_1 = 7$. Thus,

$$\vec{b} = 7\vec{a}_1 + 3\vec{a}_2 = A \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} \right).$$

Are there other vectors $\vec{x}$ with T images equal to $\vec{b}$? What does it mean to say that the T image of $\vec{x}$ is $\vec{b}$? This just means that $T(\vec{x}) = \vec{b}$; right?

So, we must find all solutions to $T(\vec{x}) = \vec{b}$, or equivalently, all solutions to $A\vec{x} = \vec{b}$.

What is the solution set for $T(\vec{x}) = \vec{b}$?

We seek <u>all</u> solutions to $T(\vec{x}) = \vec{b}$, i.e., the solution set for $A\vec{x} = \vec{b}$. The solution sets for $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{0}$ are related in a special way, each is just a translate of the other—they are parallel to each other.

We know that the solution set to $A\vec{x} = \vec{0}$ is the line $\mathbb{L} = \mathcal{S}pan\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$.

Also, $A \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = \vec{b}$; that is, $\vec{p} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$ is a particular solution to $A\vec{x} = \vec{b}$.

Therefore, the solution set for $A\vec{x} = \vec{b}$ is the line thru $\vec{p}$ that is parallel to the line $\mathbb{L}$. Algebraically, in parametric vector form, these solutions are

$$\vec{x} = \vec{p} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{where } t \text{ is any scalar.}$$

These are the only vectors $\vec{x}$ with $T(\vec{x}) = \vec{b}$. (◡̈)