Linear Transformations
Chapter 3, Section 6
An Example

Applied Linear Algebra
MATH 5112/6012

Define $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$.

- Find the domain, codomain, and range of T.
- Determine whether or not $\vec{b}$ belongs to the range of T, and if so, find all vectors $\vec{x}$ such that the T image of $\vec{x}$ is $\vec{b}$, where

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Finding the range of T requires more work.

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There are two different ways to proceed.

The range of T is $Span\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

Let's row reduce $A = \left[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\right]$.

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Let's row reduce $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix}$. Not hard to see that

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Note that $\vec{a}_1$ and $\vec{a}_2$ are the pivot columns of A, right?

Last part of problem

We now know that $\mathcal{R}ng(T)$ is the 2-plane $\mathcal{S}pan\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 1 \\ 5 \end{bmatrix} \right\}$.

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$$\vec{b} = s_1\vec{a}_1 + s_2\vec{a}_2 = s_1\begin{bmatrix}1\\3\\0\\-3\end{bmatrix} + s_2\begin{bmatrix}-2\\-4\\1\\5\end{bmatrix}?$$

Is $\vec{b}$ in $\mathcal{R}ng(T)$?

Actually, it is easy to find scalars s_1, s_2 so that

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What is the solution set for $T(\vec{x}) = \vec{b}$?

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These are the only vectors $\vec{x}$ with $T(\vec{x}) = \vec{b}$. (⌣̈)