Linear Transformations and their Standard Matrix

Applied Linear Algebra
MATH 5112/6012

UNIVERSITY OF Cincinnati

The Matrix Transformation $\vec{y} = A\vec{x}$

When A is an $m \times n$ matrix, we can define a *transformation* (aka, a *function*) from $\mathbb{R}^n$ to $\mathbb{R}^m$ via the rule $\vec{y} = T(\vec{x}) = A\vec{x}$.

Here the input variable $\vec{x}$ comes from $\mathbb{R}^n$, it gets multiplied by the matrix A via the formula

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n,$$

where $\vec{a}_j$ is the j^{th} column of A, and the resulting output is $\vec{y}$. Thus,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \vec{y} = T(\vec{x}) = A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

We write $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ (and say, T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$), meaning that $\mathbb{R}^n$ is the *domain* of T and $\mathbb{R}^m$ the *codomain*.

Linear Transformations $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

We call $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ a *linear transformation* provided

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

and

$$T(s\vec{v}) = sT(\vec{v})$$

for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ and all scalars s.

For example, every matrix transformation has these two properties.
This holds by simple properties of the matrix product $A\vec{x}$.

Which of the geometric transformations are *linear*? (Most, but not all!)

Note that for **any** LT T we *always have* $T(\vec{0}) = \vec{0}$. Right? Why?

Properties of Linear Transformations

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be a linear transformation. Then $\boxed{T(\vec{0}) = \vec{0}}$.

More importantly, T *preserves* all linear combinations; i.e., the T-image of a LC of vectors $\vec{v}_j$ is a LC of $T(\vec{v}_j)$ using the same scalars. That is,

$$T(s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p) = s_1T(\vec{v}_1) + s_2T(\vec{v}_2) + \cdots + s_pT(\vec{v}_p)$$

or more simply—using "summation" notation—

$$T\left(\sum_{j=1}^{p} s_j\vec{v}_j\right) = \sum_{j=1}^{p} s_j T(\vec{v}_j).$$

This is called the *linearity principle*.

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for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ and all scalars s.

Linear Transformations are Matrix Transformations

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A so that $T(\vec{x}) = A\vec{x}$. We call A the *standard matrix* for T.

This means that LTs and MTs are the same objects!

How do we find A? We use the linearity principle!

To start, note that each vector $\vec{x}$ can be written as a LC as follows.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

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We can write this more simply as follows. Let

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Then, $\vec{x} = \sum_{j=1}^{n} x_j \vec{e}_j$. By the LP,

$$T(\vec{x}) = T\left(\sum_{j=1}^{n} x_j \vec{e}_j\right) = \sum_{j=1}^{n} x_j T(\vec{e}_j).$$

We have $\vec{x} = \sum_{j=1}^{n} x_j \vec{e_j}$, and then by the LP,

$$T(\vec{x}) = T\left(\sum_{j=1}^{n} x_j \vec{e_j}\right) = \sum_{j=1}^{n} x_j T(\vec{e_j}).$$

Setting $\vec{a_j} = T(\vec{e_j})$ we get

$$T(\vec{x}) = \sum_{j=1}^{n} x_j \vec{a_j} = x_1 \vec{a_1} + x_2 \vec{a_2} + \cdots + x_n \vec{a_n},$$

but this means $T(\vec{x}) = A\vec{x}$ where A is the matrix with columns $\vec{a_j}$.

Linear Transformations are Matrix Transformations

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A so that $\boxed{T(\vec{x}) = A\vec{x}}$. We call A the *standard matrix* for T.

The *columns* of A are simply $\boxed{\vec{a}_j = T(\vec{e}_j)}$; these n vectors live in $\mathbb{R}^m$.

Thus $\boxed{A = \begin{bmatrix}\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_n\end{bmatrix}}$.

An Example

Suppose $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^4$ is given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - 3z \\ x + y + z \\ y - 2z \\ 3x + z \end{bmatrix}$.

What is the standard matrix for T? It's $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$.
Here

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

and so

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \\ 3 & 0 & 1 \end{bmatrix}. \quad \text{Therefore, } T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$