Linear Transformations and their Standard Matrix

Applied Linear Algebra
MATH 5112/6012

University of Cincinnati

The Matrix Transformation $\vec{y} = A\vec{x}$

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We write $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ (and say, T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$), meaning that $\mathbb{R}^n$ is the *domain* of T and $\mathbb{R}^m$ the *codomain*.

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This is called the *linearity principle*.

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but this means $T(\vec{x}) = A\vec{x}$ where A is the matrix with columns $\vec{a}_j$.

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Thus $\boxed{A = \left[\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_n\right]}$.

An Example

Suppose $\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^4$ is given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - 3z \\ x + y + z \\ y - 2z \\ 3x + z \end{bmatrix}$.

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What is the standard matrix for T? It's $A = \left[T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)\right]$.

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What is the standard matrix for T? It's $A = \left[T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)\right]$.

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$$T(\vec{e}_1) =$$

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What is the standard matrix for T? It's $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$. Here

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \\ 3 & 0 & 1 \end{bmatrix}. \quad \text{Therefore, } T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$