Dimension, Rank, Nullity

Applied Linear Algebra – MATH 5112/6012

UNIVERSITY OF Cincinnati

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The *dimension* of the trivial vector space $\{\vec{0}\}$ is defined to be 0.

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Examples

Let $\mathbb{U}^{2\times 2}$ and $\mathbb{S}^{2\times 2}$ be the spaces of all upper triangular and all symmetric 2×2 matrices, respectively. Let's find $\dim \mathbb{U}^{2\times 2}$ and $\dim \mathbb{S}^{2\times 2}$.

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What about upper triangular and symmetric $n \times n$ matrices?

$A = \left[\vec{a}_1\ \vec{a}_2\ \ \ldots\ \ \vec{a}_n\right]$ an $m \times n$ matrix and $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

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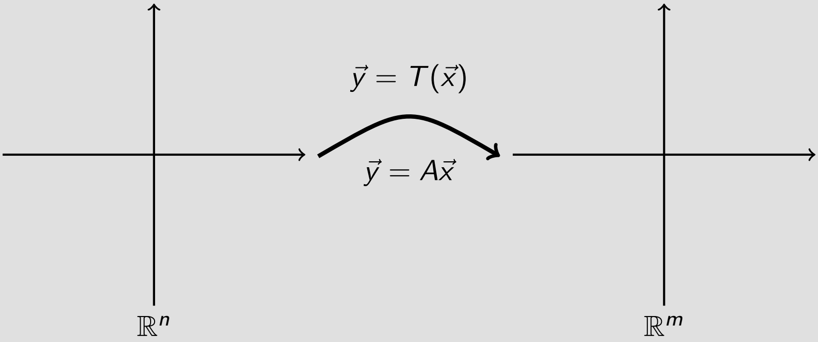

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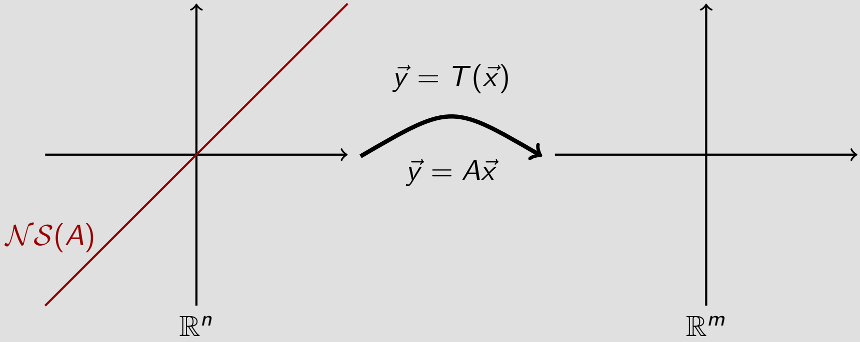

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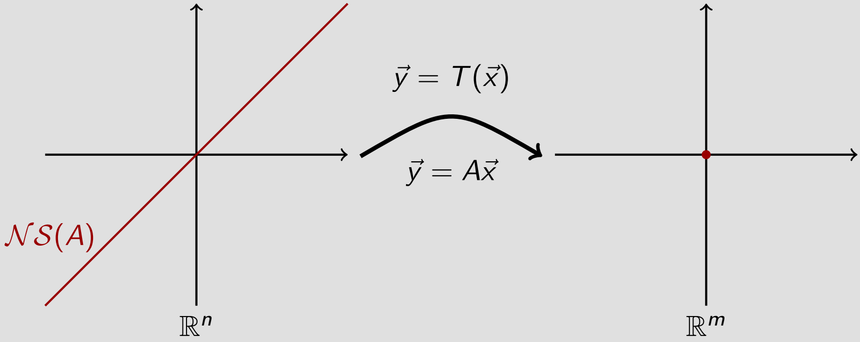

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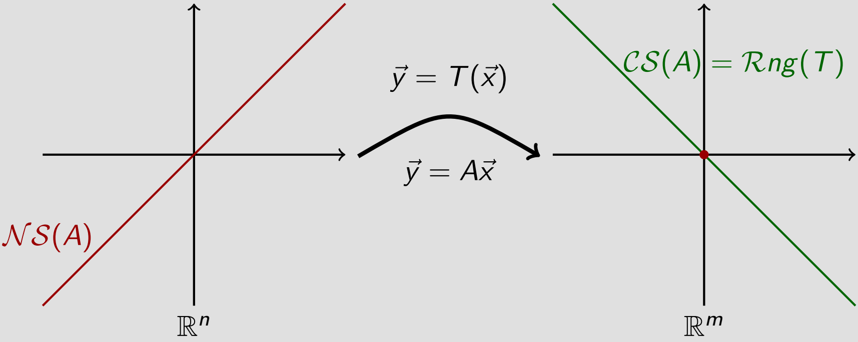

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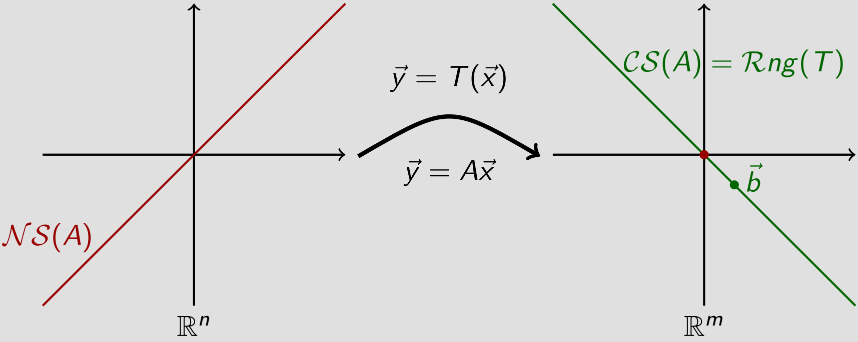

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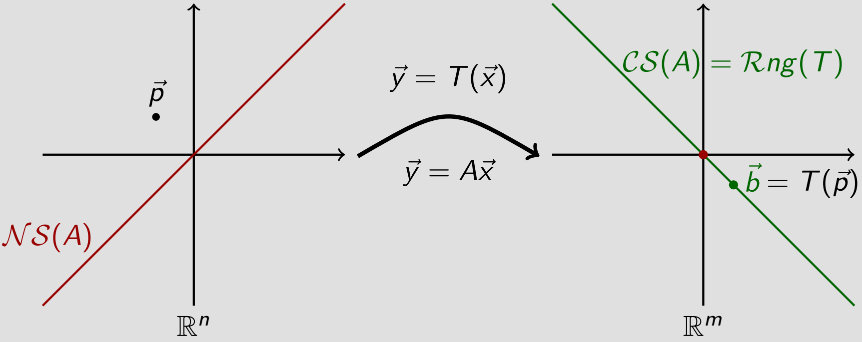

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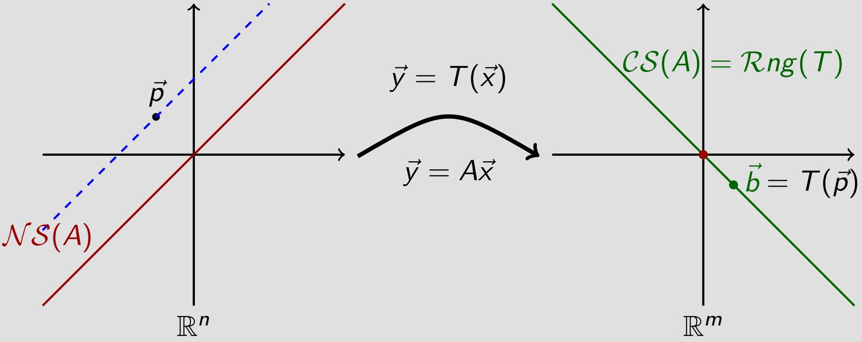

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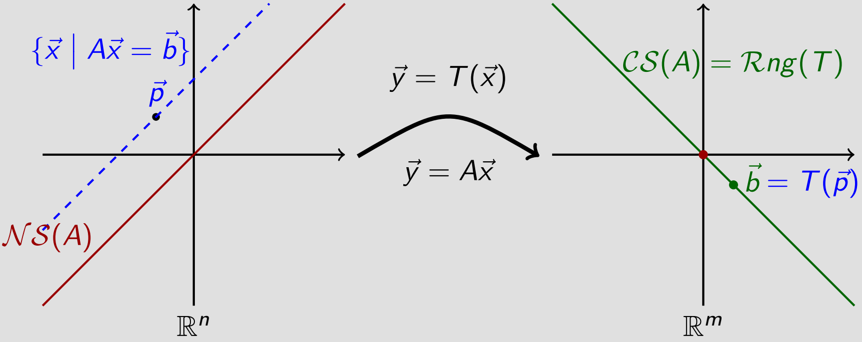

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This last fact is called the *Rank-Nullity Theorem*.

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Let $\mathbb{V}$ be a vector space.

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The Rank-Nullity Theorem helps here!

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Thus $\mathcal{NS}(A)$ is a 37-plane in $\mathbb{R}^{56}$. Remember, the solution spaces to $A\vec{x} = \vec{b}$ are all just translates of $\mathcal{NS}(A)$.

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Thus $\mathcal{NS}(A)$ is a 37-plane in $\mathbb{R}^{56}$. Remember, the solution spaces to $A\vec{x} = \vec{b}$ are all just translates of $\mathcal{NS}(A)$. Thus every solution space to $A\vec{x} = \vec{b}$ is an *affine* 37-plane in $\mathbb{R}^{56}$.