Bases and Coordinates

Linear Algebra
MATH 5112/6012

Bases

Let $\mathbb{V}$ be a vector subspace of $\mathbb{R}^n$.

Definition

A bunch of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is called a *basis* for $\mathbb{V}$ if and only if

- $\mathcal{B}$ is linearly independent, and
- $\mathcal{B}$ spans $\mathbb{V}$ (i.e., $\mathbb{V} = \mathcal{S}pan(\mathcal{B})$).

So, what are bases useful for? Why do we care about these?

First, since $\mathbb{V} = \mathcal{S}pan(\mathcal{B})$, each vector $\vec{x}$ in $\mathbb{V}$ can be written as a LC of basis vectors. That is, there are scalars scalars $c_1, c_2, \ldots, c_p$ such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \quad \left(\text{more compactly}\,,\ \vec{x} = \sum_{i=1}^{p} c_i\vec{v}_i\right).$$

Next, $\mathcal{B}$ linearly independent says this is the **only** way $\vec{x}$ can be so written.

Bases

Why is $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \sum_{i=1}^{p} c_i\vec{v}_i$ the **only** way that $\vec{x}$ can be written as a LC of vectors in the basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$?

To see this, suppose we also have $\vec{x} = \sum_{i=1}^{p} d_i\vec{v}_i$ for some scalars d_i. Then by subtracting $\vec{x} = \sum_{i=1}^{p} d_i\vec{v}_i$ from $\vec{x} = \sum_{i=1}^{p} c_i\vec{v}_i$ we get

$$\sum_{i=1}^{p}(c_i - d_i)\vec{v}_i = \sum_{i=1}^{p} c_i\vec{v}_i - \sum_{i=1}^{p} d_i\vec{v}_i = \vec{x} - \vec{x} = \vec{0}.$$

But since $\mathcal{B}$ is LI, this implies that $c_i = d_i$ for all i. Right?

Using a Basis

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis for a vector subspace $\mathbb{V}$. Then for each vector $\vec{x}$ in $\mathbb{V}$, there are *unique* scalars $c_1, c_2, \ldots, c_p$ such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \quad \left(\text{more compactly},\ \vec{x} = \sum_{i=1}^{p} c_i\vec{v}_i\right).$$

Definition

We call $c_1, c_2, \ldots, c_p$ the *coordinates of $\vec{x}$ relative to $\mathcal{B}$*.

We also call $c_1, c_2, \ldots, c_p$ the *$\mathcal{B}$-coordinates of $\vec{x}$* and $\left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$ is the *$\mathcal{B}$-coordinate vector for $\vec{x}$*.

Note that $\left[\vec{x}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

Example

Let's find a basis for the plane $\mathbb{W}$ in $\mathbb{R}^3$ given by $x + 2y + 3z = 0$. One way to do this is to recognize that $\mathbb{W} = \mathcal{NS}\big([1\ 2\ 3]\big)$ and proceed "as usual". However, it is pretty darn easy to find two LI vectors that span $\mathbb{W}$; these two vectors will form a basis for $\mathbb{W}$.

How can we find *one non-zero* vector in $\mathbb{W}$; i.e., one *non-zero* solution to $x + 2y + 3z = 0$? (We did this sort of thing on the first day of class!) Just set one variable equal to 0, one variable equal to 1, and solve for the third variable; right?

With $z = 0, y = 1$ we get $x = -2$; and with $y = 0, z = 1$ we get $x = -3$. It now follows that a basis for $\mathbb{W}$ is given by

$$\mathcal{B} = \{\vec{w}_1, \vec{w}_2\} \quad \text{where} \quad \vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \; \vec{w}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Example

So, a basis for the plane $\mathbb{W}$ (i.e. the soln set to $x + 2y + 3z = 0$) is given by

$$\mathcal{B} = \{\vec{w}_1, \vec{w}_2\} \quad \text{where} \quad \vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ \vec{w}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Thus every vector $\vec{w}$ in $\mathbb{W}$ can be written in a unique way as $\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2$ where c_1, c_2 are the $\mathcal{B}$-coordinates of $\vec{w}$, and then

$$\left[\vec{w}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{is the } \textit{$\mathcal{B}$-coordinate vector for $\vec{w}$}.$$

For example,

$$\vec{w} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} = 4\vec{w}_1 - 3\vec{w}_2 \quad \text{is in } \mathbb{W} \text{ and} \quad \left[\vec{w}\right]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Note that while $\vec{w}$ is in $\mathbb{R}^3$, $\left[\vec{w}\right]_{\mathcal{B}}$ is in $\mathbb{R}^2$.

Example—Null Space and Column Space

Find bases for the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using elem row ops, we find the indicated REF and RREF for A. Thus columns 1,2,4 are pivot columns for A, so a basis for $\mathcal{CS}(A)$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ and we see that $\mathcal{CS}(A)$ is a 3-plane in $\mathbb{R}^4$.

Let's focus on $\mathcal{NS}(A)$. So, we need to "solve" $A\vec{x} = \vec{0}$. The free variables are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Example—Null Space and Column Space

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{NS}(A)$ is a vector subspace of $\mathbb{R}^5$. To "find" $\mathcal{NS}(A)$, we solve $A\vec{x} = \vec{0}$.
Free vrbls are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Thus $A\vec{x} = \vec{0}$ iff $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s + t \\ -s + 2t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

So, $\mathcal{NS}(A)$ is a 2-plane in $\mathbb{R}^5$ and the above two vectors form a basis.

Example—Null Space and Column Space

$\mathcal{NS}(A)$, a 2-plane in $\mathbb{R}^5$, has basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

If $\vec{x}$ in $\mathcal{NS}(A)$ is given by $\vec{x} = s\vec{v}_1 + t\vec{v}_2$,
then $\left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} s \\ t \end{bmatrix}$ which is a vector in $\mathbb{R}^2$.

We can identify $\mathcal{NS}(A)$ with the *st*-plane!