Bases and Coordinates

Linear Algebra
MATH 5112/6012

UNIVERSITY OF Cincinnati

Bases

Let $\mathbb{V}$ be a vector subspace of $\mathbb{R}^n$.

Definition

A bunch of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is called a *basis* for $\mathbb{V}$ if and only if

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$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \quad \left(\text{more compactly}\,,\ \vec{x} = \sum_{i=1}^{p} c_i\vec{v}_i\right).$$

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Next, $\mathcal{B}$ linearly independent says this is the **only** way $\vec{x}$ can be so written.

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Note that $\left[\vec{x}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

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With $z = 0, y = 1$ we get $x = -2$; and with $y = 0, z = 1$ we get $x = -3$. It now follows that a basis for $\mathbb{W}$ is given by

$$\mathcal{B} = \{\vec{w}_1, \vec{w}_2\} \quad \text{where} \quad \vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ \vec{w}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

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$$\vec{w} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} = 4\vec{w}_1 - 3\vec{w}_2 \quad \text{is in } \mathbb{W} \text{ and} \quad \left[\vec{w}\right]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

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Note that while $\vec{w}$ is in $\mathbb{R}^3$, $\left[\vec{w}\right]_{\mathcal{B}}$ is in $\mathbb{R}^2$.

Example—Null Space and Column Space

Find bases for the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix}.$$

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Thus columns 1,2,4 are pivot columns for A, so a basis for $\mathcal{CS}(A)$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

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Let's focus on $\mathcal{NS}(A)$. So, we need to "solve" $A\vec{x} = \vec{0}$.

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Using elem row ops, we find the indicated REF and RREF for A.
Thus columns 1,2,4 are pivot columns for A, so a basis for $\mathcal{CS}(A)$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ and we see that $\mathcal{CS}(A)$ is a 3-plane in $\mathbb{R}^4$.
Let's focus on $\mathcal{NS}(A)$. So, we need to "solve" $A\vec{x} = \vec{0}$. The free variables are $x_3 = s$, $x_5 = t$;

Example—Null Space and Column Space

Find bases for the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using elem row ops, we find the indicated REF and RREF for A. Thus columns 1,2,4 are pivot columns for A, so a basis for $\mathcal{CS}(A)$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ and we see that $\mathcal{CS}(A)$ is a 3-plane in $\mathbb{R}^4$.

Let's focus on $\mathcal{NS}(A)$. So, we need to "solve" $A\vec{x} = \vec{0}$. The free variables are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Example—Null Space and Column Space

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{NS}(A)$ is a vector subspace of

Example—Null Space and Column Space

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$\mathcal{NS}(A)$ is a vector subspace of $\mathbb{R}^5$. To "find" $\mathcal{NS}(A)$, we solve $A\vec{x} = \vec{0}$.
Free vrbls are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Example—Null Space and Column Space

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Free vrbls are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Thus $A\vec{x} = \vec{0}$ iff $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s + t \\ -s + 2t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

Example—Null Space and Column Space

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{NS}(A)$ is a vector subspace of $\mathbb{R}^5$. To "find" $\mathcal{NS}(A)$, we solve $A\vec{x} = \vec{0}$.
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Thus $A\vec{x} = \vec{0}$ iff $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s + t \\ -s + 2t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

So, $\mathcal{NS}(A)$ is a 2-plane in $\mathbb{R}^5$ and the above two vectors form a basis.

Example—Null Space and Column Space

Example—Null Space and Column Space

$\mathcal{NS}(A)$, a 2-plane in $\mathbb{R}^5$, has basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Example—Null Space and Column Space

$\mathcal{NS}(A)$, a 2-plane in $\mathbb{R}^5$, has basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ where

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If $\vec{x}$ in $\mathcal{NS}(A)$ is given by $\vec{x} = s\vec{v}_1 + t\vec{v}_2$,

then $\left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} s \\ t \end{bmatrix}$ which is a vector in $\mathbb{R}^2$.

Example—Null Space and Column Space

$\mathcal{NS}(A)$, a 2-plane in $\mathbb{R}^5$, has basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ where

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If $\vec{x}$ in $\mathcal{NS}(A)$ is given by $\vec{x} = s\vec{v}_1 + t\vec{v}_2$,

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We can identify $\mathcal{NS}(A)$ with the *st*-plane!