Subspaces of Euclidean Space $\mathbb{R}^n$

Applied Linear Algebra
MATH 5112/6012

UNIVERSITY OF Cincinnati

What is a subspace?

Let $\mathbb{V}$ be a collection of vectors in $\mathbb{R}^n$. (For example, $\mathbb{V}$ could be a solution set to some equation, or it could be all the vectors that have third coordinate -7.)

We say that $\mathbb{V}$ *closed with respect to scalar multiplication* if and only if whenever $\vec{v}$ is in $\mathbb{V}$ and s is any scalar, then $s\vec{v}$ is also in $\mathbb{V}$. For example, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}\}$ (for some $\vec{v}$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to scalar multiplication. In fact, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ (for any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to scalar multiplication.

We say that $\mathbb{V}$ *closed with respect to vector addition* if and only if whenever $\vec{u}$ and $\vec{v}$ are in $\mathbb{V}$, then $\vec{u} + \vec{v}$ is also in $\mathbb{V}$. For example, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}\}$ (for some $\vec{v}$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to vector addition. In fact, if $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ (for any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$), then $\mathbb{V}$ is closed with respect to vector addition.

We call $\mathbb{V}$ a *vector subspace* of $\mathbb{R}^n$ if and only if ...

What is a subspace?

Let $\mathbb{V}$ be a collection of vectors in $\mathbb{R}^n$.
We call $\mathbb{V}$ a *vector subspace* of $\mathbb{R}^n$ if and only if

- $\vec{0}$ is in $\mathbb{V}$,
- $\mathbb{V}$ *closed with respect to vector addition*, and
- $\mathbb{V}$ *closed with respect to scalar multiplication*.

Some simple examples:

- $\mathbb{V} = \{\vec{0}\}$ is the *trivial* vector subspace
- $\mathbb{V} = \mathbb{R}^n$ is a vector subspace of itself (also kinda *trivial*)
- $\mathbb{V} = \mathcal{S}pan\{\vec{v}\}$ (for any $\vec{v}$ in $\mathbb{R}^n$)
- $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ (for any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$)

A simple non-example:

- $\mathbb{V} = \left\{\text{all } \vec{v} \text{ in } \mathbb{R}^4 \text{ with third coordinate -7}\right\}$ is not a subspace

More Examples—Which are, or are not, vector subspaces?

For each $\mathbb{V}$, decide whether or not $\mathbb{V}$ is closed with respect to scalar multiplication and/or closed with respect to vector addition.

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x + y = 1 \right\}$$

$\vec{0}$ not in $\mathbb{V}$, so not VSS

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, xy \geq 0 \right\}$$

$\mathbb{V}$ not closed wrt vector add

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x \geq 0 \text{ and } y \geq 0 \right\}$$

$\mathbb{V}$ not closed wrt scalar mult

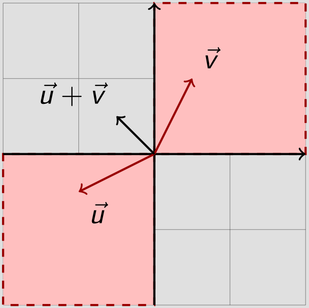

Figure: $xy \geq 0$

Vector Subspaces—Basic Example

Recall that a collection $\mathbb{V}$ of vectors (in $\mathbb{R}^n$) is a *vector subspace* (of $\mathbb{R}^n$) if and only if

- $\vec{0}$ is in $\mathbb{V}$,
- $\mathbb{V}$ *closed with respect to vector addition* ($\vec{u}, \vec{v}$ in $\mathbb{V} \implies \vec{u} + \vec{v}$ in $\mathbb{V}$)
- $\mathbb{V}$ *closed with respect to scalar mult* (s scalar , $\vec{v}$ in $\mathbb{V} \implies s\vec{v}$ in $\mathbb{V}$)

Let $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$. Let's show that $\mathbb{V}$ is closed wrt vector addition. Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{V}$. This means there are scalars $s_1, s_2, \dots, s_p$ and $t_1, t_2, \dots, t_p$ with

$$\vec{u} = s_1\vec{v}_1 + \cdots + s_p\vec{v}_p \quad \text{and} \quad \vec{v} = t_1\vec{v}_1 + \cdots + t_p\vec{v}_p$$

so

$$\vec{u} + \vec{v} = (s_1 + t_1)\vec{v}_1 + (s_2 + t_2)\vec{v}_2 + \cdots + (s_p + t_p)\vec{v}_p$$

which is a vector in $\mathbb{V}$.

Homework: Show that $\mathbb{V}$ is closed wrt scalar multiplication.

Vector Subspaces—Basic Example

Just saw that any $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is both closed wrt vector addition and closed wrt scalar multiplication.

Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a vector subspace.

In fact, *every* vector subspace can be expressed this way!

Example (Column Space of a Matrix)

The *column space* $\mathcal{CS}(A)$ of a matrix A is the span of the columns of A. Thus is A is an $m \times n$ matrix, then $\mathcal{CS}(A)$ is a VSS of $\mathbb{R}^m$.

If $A = \left[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n\right]$, then $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.

Column Space of a Matrix

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix; so, each $\vec{a}_j$ is in $\mathbb{R}^m$.

The *column space* $\mathcal{CS}(A)$ of A is the span of the columns of A, i.e., $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$.

Three Ways to View $\mathcal{CS}(A)$

The column space $\mathcal{CS}(A)$ of $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ is:

- $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$
- $\mathcal{CS}(A) = \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\}$
- $\mathcal{CS}(A) = \mathcal{R}ng(T)$ where $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

Three Ways to View the Column Space $\mathcal{CS}(A)$

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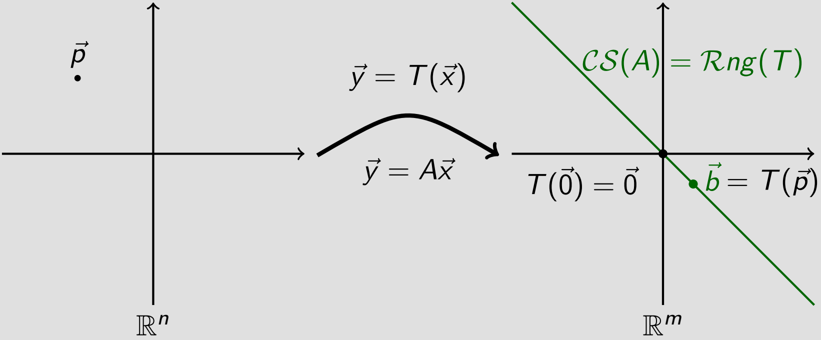
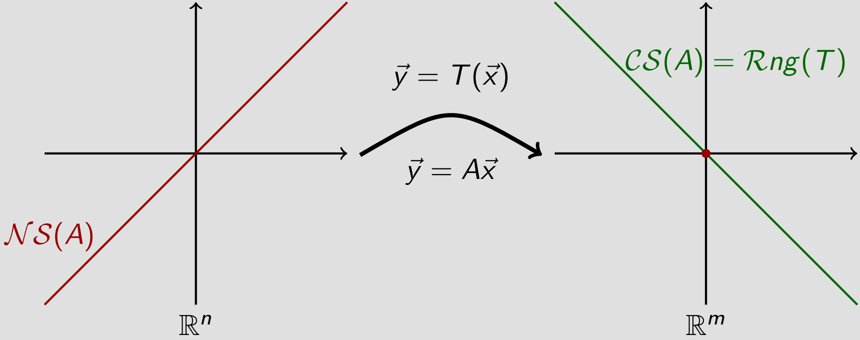


Vector Subspaces—Another Basic Example

Again, let A be an $m \times n$ matrix. The *null space* $\mathcal{NS}(A)$ of A is

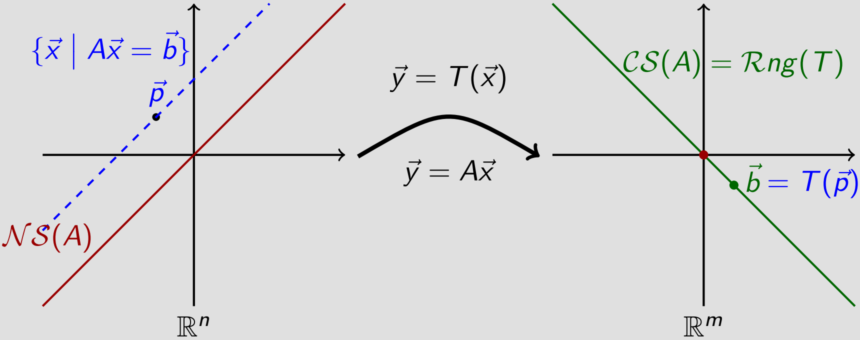
$$\mathcal{NS}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\};$$

just the solution set for the homogeneous equation $A\vec{x} = \vec{0}$. This is a vector subspace of $\mathbb{R}^n$.



$A = \left[\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n\right]$ an $m \times n$ matrix and $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

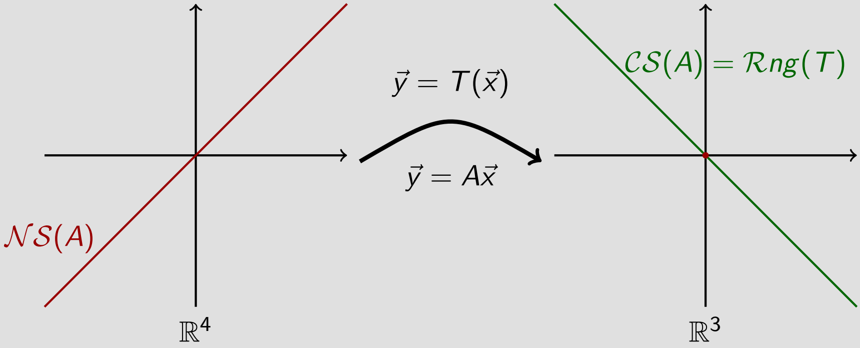
$$\begin{aligned} \mathcal{NS}(A) &= \{\vec{x} \mid A\vec{x} = \vec{0}\} \quad \text{and} \\ \mathcal{CS}(A) &= \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \\ &= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\} \\ &= \mathcal{CS}(A) = \mathcal{R}ng(T) \end{aligned}$$



Null Space and Column Space Example

Find the null space and column space of

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix}$$



What should we do now? How about row reducing A?

Vector Subspaces—Basic Fact

Recall that a collection $\mathbb{V}$ of vectors (in $\mathbb{R}^n$) is a *vector subspace* (of $\mathbb{R}^n$) if and only if

- $\vec{0}$ is in $\mathbb{V}$,
- $\mathbb{V}$ *closed with respect to vector addition* ($\vec{u}, \vec{v}$ in $\mathbb{V} \implies \vec{u} + \vec{v}$ in $\mathbb{V}$)
- $\mathbb{V}$ *closed with respect to scalar mult* (s scalar , $\vec{v}$ in $\mathbb{V} \implies s\vec{v}$ in $\mathbb{V}$)

If $\mathbb{V}$ is a vector subspace; $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{V}$; $s_1, s_2, \ldots, s_p$ are scalars: then $s_1\vec{v}_1, s_2\vec{v}_2, \ldots, s_p\vec{v}_p$ all in $\mathbb{V}$, so $s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p$ is in $\mathbb{V}$.

Any LC of vectors in a VSS $\mathbb{V}$ is a vector in $\mathbb{V}$!

Basic Fact about Vector SubSpaces

Let $\mathbb{V}$ be a vector subspace. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are in $\mathbb{V}$.
Then $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ lies in $\mathbb{V}$.