The Inverse of a Matrix

Applied Linear Algebra
Math 5112/6012

Invertible Matrices

An $n \times n$ matrix A is *invertible* if and only if there is another $n \times n$ matrix C with $\boxed{A\,C = I = C\,A}$, where I is $n \times n$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Some matrices are not invertible: $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$ is not invertible.

How can we tell when a matrix is invertible?

How can we find such a matrix C?

Invertible Matrices

An $n \times n$ matrix A is *invertible* if and only if there is another $n \times n$ matrix C with $\boxed{A\,C = I = C\,A}$.

When this holds, there is only ONE such matrix C.

Suppose we have $C_1 A = I = A C_2$. Then

$$C_1 = C_1 I = C_1(A\,C_2) = (C_1 A)C_2 = I\,C_2 = C_2$$

so $C_1 = C_2$.

We call C the *inverse* of A and write $A^{-1} = C$.

Remember, not all matrices have an inverse.

Example

Remember, for numbers $x^{-1} = x \implies x^2 = 1$, so $x = \pm 1$.

Look at

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So, above is an example of an A with $A^{-1} = A$, and $A \neq \pm I$.

Example

Look at

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so,

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$$

So, above is another example of an A with $A^{-1} = A$, and $A \neq \pm I$.

The significance of $AC = I$ for a given A

Suppose there is an $n \times n$ matrix C with $AC = I$.

Look at the equation $A\vec{x} = \vec{b}$ for some given rhs $\vec{b}$.

Try $\vec{x} = C\vec{b}$. Get

$$A\vec{x} = AC\vec{b} = I\vec{b} = \vec{b}.$$

When there is C with $AC = I$, we *know* that for *any* rhs $\vec{b}$, $A\vec{x} = \vec{b}$ has a solution, namely $\vec{x} = C\vec{b}$. Solutions always exist for *any* rhs $\vec{b}$!

What does this say about a REF for A?

What does this say about the columns of A?

The significance of $CA = I$ for a given A

Suppose there is an $n \times n$ matrix C with $CA = I$.

Look at the equation $A\vec{x} = \vec{b}$ for some given rhs $\vec{b}$.

Suppose that $\vec{p}$ is some solution, so $A\vec{p} = \vec{b}$. Multiply both sides by C to get

$$\vec{p} = I\vec{p} = CA\vec{p} = C\vec{b}.$$

When there is C with $CA = I$, we *know* that $A\vec{x} = \vec{b}$ has the *unique* solution, $\vec{x} = C\vec{b}$. If there is a solution (for some rhs $\vec{b}$), then the solution is unique!

What does this say about a REF for A?

What does this say about the columns of A?

Finding C (i.e., finding A^{-1})

First, A may be non-invertible, right? There may be no such C.

Looking for C with $AC = I$. Think about a variable $n \times n$ matrix X. Trying to solve $AX = I$. Look at the columns of each side.

Next, $\mathrm{Col}_j(AX) = A\ \mathrm{Col}_j(X)$, so we're trying to solve

$$AX = I \quad \text{or} \quad \mathrm{Col}_j(AX) = \mathrm{Col}_j(I) \quad \text{or} \quad A\vec{x} = \vec{e}_j$$

where $\vec{x} = \mathrm{Col}_j(X)$, and we solve for each $1 \leq j \leq n$.

Look at super-sized augmented matrix $\left[A \vdots I\right]$. Put into *reduced* REF.

Do elem row ops to get $\left[A \vdots I\right] \xrightarrow[\text{reduced REF}]{\text{row reduce to}} \left[E \vdots F\right]$. Two possibilities:

If $E \neq I$, then A not invertible.

If $E = I$, then $F = A^{-1}$.

Finding A^{-1}, if it exists

Look at super-sized augmented matrix $\left[A \vdots I\right]$. Put into *reduced* REF.

Do elementary row operations to get $\left[A \vdots I\right] \xrightarrow[\text{reduced REF}]{\text{row reduce to}} \left[E \vdots F\right]$.

Get two possibilities:

If $E \neq I$, then A not invertible.

If $E = I$, then $F = A^{-1}$.

Example

Determine if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is invertible, and find A^{-1} if it exists.

Gotta row reduce

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

$$\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&1&2&3&0&1&0&0\\1&1&1&2&0&0&1&0\\1&1&1&1&0&0&0&1\end{array}\right]\xrightarrow{R_4-R_3}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&1&2&3&0&1&0&0\\1&1&1&2&0&0&1&0\\0&0&0&-1&0&0&-1&1\end{array}\right]$$

$$\xrightarrow{R_3-R_2}\xrightarrow{-R_4}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&1&2&3&0&1&0&0\\0&0&-1&-1&0&-1&1&0\\0&0&0&1&0&0&1&-1\end{array}\right]\xrightarrow[-R_3]{R_3+R_4}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&1&2&3&0&1&0&0\\0&0&1&0&0&1&-2&1\\0&0&0&1&0&0&1&-1\end{array}\right]$$

$$\xrightarrow{R_2-R_1}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&-1&-1&-1&-1&1&0&0\\0&0&1&0&0&1&-2&1\\0&0&0&1&0&0&1&-1\end{array}\right]\xrightarrow[-R_2]{R_2+R_3+R_4}$$

$$\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&1&0&0&1&-2&1&0\\0&0&1&0&0&1&-2&1\\0&0&0&1&0&0&1&-1\end{array}\right]\xrightarrow[-3R_3-4R_4]{R_1-2R_2}\left[\begin{array}{cccc|cccc}1&0&0&0&-1&1&0&1\\0&1&0&0&1&-2&1&0\\0&0&1&0&0&1&-2&1\\0&0&0&1&0&0&1&-1\end{array}\right]$$

Example

So, $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is invertible, and $A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Should check that

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is not hard to do, right?