Matrix Arithmetic

Applied Linear Algebra
MATH 5112/6012

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Matrix Notation

Suppose A is the $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$.

We can express this by writing $A = [a_{ij}]$; here $1 \leq i \leq m$ and $1 \leq j \leq n$.

Alternatively, we write $[A]_{ij} = a_{ij}$ to indicate that a_{ij} is the entry of A in the i^{th} row and j^{th} column; more briefly, a_{ij} is the i, j entry of A.

Scalar Multiplication

A matrix A can be multiplied by any scalar s to get a new matrix sA. Here sA is obtained by multiplying every entry of A by s.

Thus if $A = [a_{ij}]$, then $sA = [s\, a_{ij}]$.

Alternatively, $[sA]_{ij} = s[A]_{ij}$.

Matrix Addition

Any two matrices of the *same dimensions* can be added, entry by entry.

Let A and B both be $m \times n$ matices. Then $A + B$ is the $m \times n$ matix with $[A + B]_{ij} = [A]_{ij} + [B]_{ij}$.

That is, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A + B = [a_{ij} + b_{ij}]$.

Example

Calculate

$$2\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & 1\end{bmatrix} - \begin{bmatrix}2 & 5 & 5\\ -3 & 0 & 2\end{bmatrix} = \begin{bmatrix}2 & 4 & 6\\ 0 & -2 & 2\end{bmatrix} + \begin{bmatrix}-2 & -5 & -5\\ 3 & 0 & -2\end{bmatrix}$$

$$= \begin{bmatrix}2-2 & 4-5 & 6-5\\ 0+3 & -2 & 2-2\end{bmatrix} = \begin{bmatrix}0 & -1 & 1\\ 3 & -2 & 0\end{bmatrix}.$$

The Transpose of a Matrix

The *transpose* A^{T} of a matrix A is given by "reflecting A across its main diagonal". The rows (columns) of A become the columns (rows) of A^T.

More precisely, $[A^{\mathsf{T}}]_{ij} = [A]_{ji}$; equivalently, if $A = [a_{ij}]$, then $A^{\mathsf{T}} = [a_{ji}]$.
For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\vec{x}^{\mathsf{T}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$.

If A is square and $A^{\mathsf{T}} = A$, then A is called a *symmetric* matrix.

Matrix Multiplication

Recall that when A is an $m \times n$ matrix and $\vec{x}$ a vector in $\mathbb{R}^n$,

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

where $\vec{a}_j$ is the j^{th} column of A. Note that the number of columns in A, n, must equal the number of rows (coordinates) in $\vec{x}$.

Now we describe the general matrix product $A \cdot B$.

Caution: AB is not defined for all pairs of two matrices.

The matrix product AB is only defined when

the # of columns of A **equals** the # of rows of B.

Row Vector Times a Column Vector

Suppose $\mathbf{a}$ is the row vector (i.e., a $1 \times n$ matrix)

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

and $\vec{b}$ is the column vector (i.e., an $n \times 1$ matrix) $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

The product $\mathbf{a}\,\vec{b}$ is defined by

$$\mathbf{a}\,\vec{b} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k.$$

This agrees with our definition of $A\vec{x}$, right?

Rows and Columns of a Matrix

Let A be an $m \times n$ matrix, say $A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & & a_{mj} & \dots & a_{mn} \end{bmatrix}$.

The j^{th} column of A is the vector

$$\text{Col}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}; \quad \text{here } 1 \leq j \leq n.$$

The i^{th} row of A is the row vector

$$\text{Row}_i(A) = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}; \quad \text{here } 1 \leq i \leq m.$$

The Matrix Product AB

Caution: AB is not defined for all pairs of two matrices.

The matrix product AB is only defined when

the # of columns of A **equals** the # of rows of B.

Let A be an $m \times n$ matrix, and B be an $n \times p$ matrix.

The i, j entry of AB is simply $\boxed{[AB]_{ij} = \mathrm{Row}_i(A)\ \mathrm{Col}_j(B)}$. Here $1 \le i \le m$ and $1 \le j \le p$, so AB is an $m \times p$ matrix.

Note that if $A = [a_{ij}]$ and $B = [b_{ij}]$ (careful!), then

$$[AB]_{ij} = \mathrm{Row}_i(A)\ \mathrm{Col}_j(B) = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Some Useful Formulas

Notice that

$$A\,B = \big[A\ \mathrm{Col}_1(B)\ A\ \mathrm{Col}_2(B) \dots A\ \mathrm{Col}_n(B)\big].$$

That is, $\boxed{\mathrm{Col}_j\big(AB\big) = A\ \mathrm{Col}_j(B)}$.

We also have $\boxed{\mathrm{Row}_i\big(AB\big) = \mathrm{Row}_i(A)\,B}$.

In addition, $\boxed{\mathrm{Col}_j(A) = A\vec{e}_j}$ and $\boxed{\mathrm{Row}_i(A) = \vec{e}_i^{\,\mathsf{T}} A}$.

Checking that these formulas are valid is a good exercise to see how well you understand matrix products!

Examples

Calculate

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ -3 & 0 & 2 \end{bmatrix} \tag{2}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \tag{3}$$

$$BA = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

Matrix Multiplication Anomalies

Think about

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$$

versus

$$BA = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We see that sometimes $AB \neq BA$.
Matrix multiplication is **not** commutative; order matters.

Also, $AB = \mathbf{0}$ does **not** mean that one of A or B is $\mathbf{0}$.

What is the $\mathbf{0}$ matrix anyway?

Special Matrices

Any matrix all of whose entries are the number 0 is called a *zero matrix*. We write $\mathbf{0}$ for a zero matrix; these can be of *any* size.

Clearly, for any scalar s and any zero matrix $\mathbf{0}$, $s\mathbf{0} = \mathbf{0}$. Also, for any matrix A, $A + \mathbf{0} = A$ (provided A and $\mathbf{0}$ are of the same dimensions!). Thus $\mathbf{0}$ is the *additive identity* for matrix arithmetic.

Is there a *multiplicative identity* for matrix arithmetic?

A *square* matrix is one that has the same number of rows and columns. Note that AB and BA are both defined if and only if A and B are both square matrices with exactly the same size.

Special Matrices

The *diagonal entries* of an $n \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, \ldots, a_{nn}$. We call A a *diagonal matrix* if its non-diagonal entries are all zeroes.

The $n \times n$ diagonal matrix whose diagonal entries are all ones is called the $n \times n$ *identity matrix*; we write

$$I = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

You should check that for any square matrix A, we have both

$$A\,I = A \quad \text{and} \quad I\,A = A.$$

These I's are the *multiplicative identities* for matrix arithmetic. If A is any square matrix, can we find another square matrix C so that $AC = I$?