Rotation about the line $x = y = z$

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Lets look at a picture!

The action of T

$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^3$ is 60° rotation about the line $\mathbb{L} = \mathcal{S}pan\{\vec{m}\}$ where $\vec{m} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

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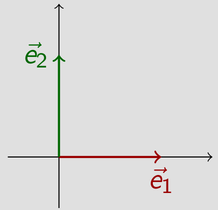
Recall that we know how to find the standard matrix for a rotation of $\mathbb{R}^2$.

Rotations of $\mathbb{R}^2$

Again, that the columns of the standard matrix for a linear transformation are given by the images of the standard basis vectors.



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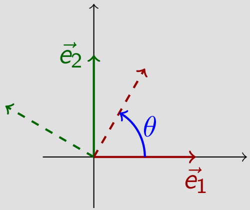
Let's rotate $\mathbb{R}^2$ by θ radians (in the ccw direction). Where does $\vec{e}_1$ go?
Note that its length does not get changed.



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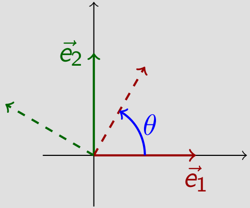

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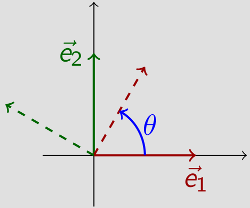

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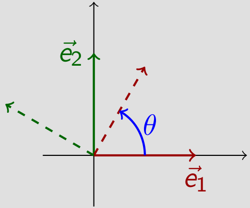

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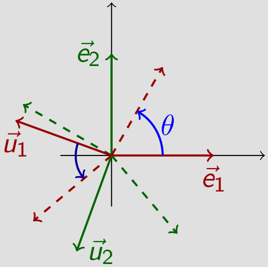

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That is, given $\vec{x}$ in $\mathbb{L}^\perp$, we multiply the coordinates of $\vec{x}$ by the above matrix to get the coordinates of $T(\vec{x})$.

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Thus, if we have a right-handed orthonormal basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for $\mathbb{R}^3$ such that $\{\vec{u}_1, \vec{u}_2\}$ is a basis for $\mathbb{L}^\perp$, then we can find the $\mathcal{U}$-matrix for T!

An orthogonal basis for $\mathbb{R}^3$–part I

We seek a right-handed orthonormal basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for $\mathbb{R}^3$ such that $\{\vec{u}_1, \vec{u}_2\}$ is a basis for $\mathbb{L}^\perp$.

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One basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{L}^\perp$ is given by using

$$\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

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To get an orthormal basis, we do the "usual". Put $\vec{u}_1 = \vec{v}_1 / \|\vec{v}_1\|$ and $\vec{u}_2 = \vec{v}_2 / \|\vec{v}_2\|$ where $\vec{v}_1 = \vec{b}_1$ and

$$\vec{v}_2 = \vec{b}_2 - \mathrm{Proj}_{\vec{v}_1}(\vec{b}_2).$$

An orthogonal basis for $\mathbb{R}^3$–part II

With $\vec{v_1} = \vec{b_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ we compute

$$\text{Proj}_{\vec{v_1}}(\vec{b_2}) = \frac{\vec{b_2} \cdot \vec{b_1}}{\vec{b_1} \cdot \vec{b_1}} \vec{b_1} = \frac{1}{2} \vec{b_1}$$

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Thus, $\vec{u_1} = \dfrac{\vec{v_1}}{\|\vec{v_1}\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{u_2} = \dfrac{\vec{v_2}}{\|\vec{v_2}\|} = \dfrac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$

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$$[T]_{\mathcal{U}} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

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A formula for a 60° rotation about the line $x = y = z$

Finally, since $\mathcal{U}$ is orthonormal, $P^{-1} = P^{\mathsf{T}}$ and so we compute

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$$
\begin{aligned}
A = [T]_{\mathcal{E}} &= P[T]_{\mathcal{U}}P^{-1} = P[T]_{\mathcal{U}}P^{\mathsf{T}} \\
&= \frac{1}{12}\begin{bmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix} \\
&= \cdots \\
&= \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}. \quad \text{(Note that } A \text{ is orthogonal!)}
\end{aligned}
$$

A formula for a 60° rotation about the line $x = y = z$

Finally, since $\mathcal{U}$ is orthonormal, $P^{-1} = P^{\mathsf{T}}$ and so we compute

$$
\begin{aligned}
A = [T]_{\mathcal{E}} &= P[T]_{\mathcal{U}}P^{-1} = P[T]_{\mathcal{U}}P^{\mathsf{T}} \\
&= \frac{1}{12}\begin{bmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix} \\
&= \cdots \\
&= \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}. \quad \text{(Note that } A \text{ is orthogonal!)}
\end{aligned}
$$

As an example, we determine the rotation of my favorite vector,

$$
T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.
$$