Systems of Linear Equations

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Linear Equations and their Solutions

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A *solution* to the above linear equation is a list $(s_1, s_2, \dots, s_n)$ of numbers such that setting $x_1 = s_1, \dots, x_n = s_n$ makes the equation a true statement. The *solution set* consists of **all** of the solutions.

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A *geometric description* for the solution set provides a pictorial visualization that represents every single solution.

Euclidean Space

The set $\mathbb{R}^2 = \{(x, y) \mid x, y \text{ any numbers}\}$ is called 2-*dimensional Euclidean space*; the *xy*-plane is a *geometric visualization* for $\mathbb{R}^2$ and allows us to draw pictures that describe certain subsets of $\mathbb{R}^2$.

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The set $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \text{ any numbers}\}$ is called 3-*dimensional Euclidean space*; *xyz*-space is a *geometric visualization* for $\mathbb{R}^3$ and allows us to draw pictures that describe certain subsets of $\mathbb{R}^2$.

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Notice that the solution set to an SLE with n variables is a subset of $\mathbb{R}^n$.

Solution Set for One Linear Equation

Again, a *solution* to the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a list $(s_1, s_2, \ldots, s_n)$ of numbers such that setting $x_1 = s_1, \ldots, x_n = s_n$ makes the equation true. We want to "find" the *solution set* which consists of **all** of the solutions.

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Hyperplanes in $\mathbb{R}^n$

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When $n = 4$ we get $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$ which gives a hyperplane in $\mathbb{R}^4$.

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