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LINEAR ALGEBRA Exam 3 Review

I Systems of Equations

(1) General System of Linear of Equations
- What is an SLE?
- What constitutes a solution?
- What are the: coefficients? variables? constants?
- What are corresponding *vector* and *matrix* equations?

(2) Geometric Viewpoint—lines, planes in $\mathbb{R}^3$ and k-planes in $\mathbb{R}^n$

(3) Solution Trichotomy—none, a unique one, infinitely many

(4) Finding Solutions to $A\vec{x} = \vec{b}$
- manipulation phase into upper triangular system
- row reduction via elementary row operations to REF
- back-substitution to get solutions, or use reduced REF
- Gaussian elimination (GE) and Gauss-Jordan elimination (GJE)
- what is general solution? how to find?

(5) Row Equivalent Systems
- why do these have the same solutions?

(6) Homogeneous Systems $A\vec{x} = \vec{0}$
- why is there always a solution?
- what is general solution? how to find?
- what is connection to non-homogeneous system?
- what is connection with *linear independence*?

(7) Existence and Uniqueness of Solutions to $A\vec{x} = \vec{b}$
- when do solutions exist? when unique?
- connection with *linear combinations*?
- connection with *linear independence*?
- how to find all $\vec{b}$ so consistent?

II Matrices

(1) Notation
- entries, columns, rows

(2) Arithmetic
- scalar multiplication and addition
- matrix multiplication
- inverses (what? how get? "big theorem")

(3) Connection with Linear Systems of Equations
- coefficient and augmented matrices
- elementary row operations
- row equivalent matrices
- GE and row echelon form (REF)
- GJE and reduced row echelon form (RREF)

(4) Interpretation of (R)REF: Existence & Uniqueness of Solutions
- the BIG theorem
- zero rows (consistency—existence of solutions)
- number of columns w/o row leader ("free" variables—uniqueness of solutions)
- number of row leaders (what?)

(5) Invertible (or Non-Singular) Matrices
- definition/meaning; connection with inverses
- how to calculate?
- connection with solutions to SLEs
- basic properties
- the BIG theorem

(6) Transpose
- definition/meaning
- symmetric and skew-symmetric matrices
- upper and lower triangular matrices
- diagonal matrices

(7) Determinants
- What is one of these? How to calculate?
- What are basic properties?

III Euclidean Space

(1) $\mathbb{R}^n$
- arithmetic (scalar multiplication, vector addition)
- scalars vs vectors vs sets of vectors

(2) Sets of Vectors
- closed wrt scalar multiplication
- closed wrt vector addition

(3) Vector subspaces of $\mathbb{R}^n$
- definition (closed wrt scalar mult and vector add)
- vector subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$
- basic example: the *null space* $\mathcal{NS}(A)$ of a matrix A
- what does $\mathcal{NS}(A) = \mathbb{R}^n$ mean?
- what does $\mathcal{NS}(A) = \{\vec{0}\}$ mean?

(4) Linear Combinations
- definition/meaning
- how to tell if a vector is a LC of other vectors?
- reinterpretation of vector subspace definition
  ($\mathbb{V}$ is a vector subspace iff it is closed wrt LC)
- interpretation of matrix multiplication
  ($A\vec{x}$ is a LC of the columns of A)

(5) Span of a Set of Vectors
- definition/meaning
- how to tell if a vector is in the span of other vectors?
- basic example: the *column space* $\mathcal{CS}(A)$ of a matrix A
- **3** interpretations of $\mathcal{CS}(A)$
- what does $\mathcal{CS}(A) = \mathbb{R}^m$ mean?
- what does $\mathcal{CS}(A) = \{0\}$ mean?

(6) Linear Independence vs Linear Dependence
- definition/meaning (especially for 2 or 3 vectors)
- how to tell when vectors LI?
- how to find LI subset of set of vectors?
- connection with solutions to homogeneous system of equations
- the BIG theorem

(7) Bases & Dimension
- definitions/meanings
- minimal spanning set vs maximal linearly independent set
- how to find (especially for $\mathcal{CS}(A)$ and $\mathcal{NS}(A)$)

- special case when get spanning set iff linearly independent

(8) Rank and Nullity
- definitions
- connection with existence and uniqueness of solutions
  what does it mean if
  the rank of A is: 0? m? $< m$?
  the nullity of A is: 0? n? > 0?
- Rank-Nullity Theorem
- what does this say about existence and uniqueness of solutions especially when $m > n$ or $m < n$ or $m = n$?
- why is the rank of A^T the same as the rank of A?

(9) Coordinates
- definition/geometric meaning
- coordinate vector
- how to find coordinates?
- change of basis coordinate change matrix
  what is it? how to find it? properties
  how to use to find coordinates?

IV Vector Spaces and Linear Transformations

(1) Vector Spaces and Subspaces
- What is a vector space?
- What is a vector subspace?
- Exs: $\mathbb{P}$, $\mathbb{P}_n$, $\mathbb{F}$

(2) Linear Combinations, Span, Linear Independence
- What is $\mathcal{S}pan\{\vec{v}_1, \ldots, \vec{v}_p\}$?
- Why is this always a vector subspace?
- When is $\{\vec{v}_1, \ldots, \vec{v}_p\}$ LI?

(3) Bases, Coordinates, Dimension
- definitions/meanings of these?
- minimal spanning set vs maximal linearly independent set
- how to find bases, coordinates, coordinate vectors?

(4) Linear Transformations
- What is one of these?
- What do the following words mean?
  domain, codomain, image (2 meanings), range, kernel
  onto, one-to-one, pre-image (2 meanings)

(5) Images and Pre-Images
- What are these?
- How can you calculate these?
- What can you say about the image of a vector subspace?

(6) Rank-Nullity Theorem
- What does this tell us about the range and kernel?
- What does this tell us about whether or not a LT is one-to-one or onto?

(7) Affine SubSpaces
- How do you translate a set of vectors?
- What is a p-dimensional affine subspace?
- How do you calculate the image or pre-image of one of these?
- What is the connection with solutions to an LSE?

(8) Matrix Representation of a Linear Transformation $\mathbb{R}^n \to \mathbb{R}^m$
- What is the standard matrix representative for a LT? How do you calculate it?
- Other matrix representatives? How do you calculate?

- $[T]_{\mathcal{BA}}$
  - What is it?
  - What does it "do"?
  - What are its columns?
  - How can you find it?
- $[T]_{\mathcal{DC}} = P_{\mathcal{DB}}[T]_{\mathcal{BA}}P_{\mathcal{AC}}$
- $[T]_{\mathcal{B}}$ (When does this make sense?)

V Eigen Stuff & Orthogonalization

(1) Eigen Stuff §§5.1, 5.2
- What are eigenvalues? eigenvectors? eigenspaces? eigenbases?
- What is a characteristic polynomial? characteristic equation?
- How do you find all these?
- What are algebraic versus geometric multiplicities and why important?

(2) Diagonalization §§5.3, 5.4
- When is a LT T diagonalizable?
- What does this tell you about the action of T?
- When is a matrix diagonalizable?
- What is connection with eigenstuff?

(3) Inner (or Scalar or Dot) Product §6.1
- definition – algebraic and geometric
- connection with angles and length
- properties

(4) Orthogonality §6.2
- $\vec{x} \perp \vec{y}$ iff ..., $\mathbb{V} \perp \mathbb{W}$ iff ...
- orthogonal *set* of vectors (LI or LD?)
- orthonormal *set* of vectors (LI or LD?)

(5) Orthogonal Complement §6.3
- definition and properties of $\mathbb{V}^{\perp}$
- how to determine $\mathbb{V}^{\perp}$
- $\dim \mathbb{V}^{\perp} = ?$
- Why is $\mathbb{R}^n = \mathbb{V} + \mathbb{V}^{\perp}$? Why is $\mathbb{R}^n = \mathbb{V} \oplus \mathbb{V}^{\perp}$?

(6) Vector Projection & Orthogonal Projection §6.3
- definitions & properties
- how to calculate
- geometric interpretation
- closest vector (aka, best approximation) idea
- distance to a vector subspace

(7) Gram-Schmidt Procedure §6.4
- what it does, how to do it, what it tells you

(8) Symmetric matrices §7.1
- When is a matrix symmetric? Why are they 'special'?
- How many "ways" can we diagonalize a symmetric matrix?

(9) Quadratic Functions §7.2
- What are these?
- What are the connections with matrices?
- What are the level sets of a quadratic function?