Contents

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COMPLEX ANALYSIS NOTES

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1. Elementary Things

1.A. **The Beginning.** What are complex numbers, and what can we do with them?

What is $\mathbb{C}$? From an informal algebraic point of view, $\mathbb{C}$ is the *field* of complex numbers $z = x + iy$, $w = u + iv$, $\zeta = \xi + i\eta$ (etc.) where the laws of complex arithmetic hold. Here $u, v, x, y, \xi, \eta \in \mathbb{R}$ are real numbers and $i^2 = -1$. Unlike the field $\mathbb{R}$ of real numbers, $\mathbb{C}$ is not an ordered field. Of course, in order to have a field there must exist both additive and multiplicative inverses.

Example: For $z \neq 0$, $z^{-1} = 1/z = \bar{z}/|z|^2$. Here $\bar{z}$ is the *complex conjugate* of z and $|z|$ is the *absolute value* (or *modulus*) of z; when $z = x + iy$, $\bar{z} := x - iy$ and $|z| := \sqrt{x^2 + y^2}$. The map $w = 1/z$ is called *complex inversion*. Notice that $z\bar{z} = |z|^2$.

Here is some more notation: when $z = x + iy$, $\Re(z) := x$ and $\Im(z) := y$ are called the *real part* and *imaginary part* of z (respectively). A complex number z is *purely real* if $\Im(z) = 0$ and is *purely imaginary* when $\Re(z) = 0$. We identify the real number field $\mathbb{R}$ with $\{z \in \mathbb{C} : \Im(z) = 0\}$ and write $i\mathbb{R}$ for $\{z \in \mathbb{C} : \Re(z) = 0\}$. Note the useful inequalities

$$|\Re(z)| \leq |z| \quad \text{and} \quad |\Im(z)| \leq |z|.$$

From a geometric point of view, we can picture $\mathbb{C}$ as a Euclidean plane by using cartesian coordinates: the complex number $z = x + iy$ is identified with the point (x, y). In this setting the x-axis is called the *real axis*, which is just $\mathbb{R} \subset \mathbb{C}$, and the y-axis, the so-called *imaginary axis*, is $i\mathbb{R} \subset \mathbb{C}$. Notice that $\bar{z}$ is the reflection of z across the real axis and $|z|$ is the distance from z to the origin. Three other sets which are worthy of special designations are the *unit disk* $\mathbb{D} := \{z : |z| < 1\}$, the *unit circle* $\mathbb{T} := \{z : |z| = 1\}$, and the *right half-plane* $\mathbb{H} := \{z : \Re(z) > 0\}$. Also, $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ and $\mathbb{D}_* := \mathbb{D} \setminus \{0\}$ are the punctured plane and punctured disk.

Example: Let $w = T(z) = \dfrac{1-z}{1+z} = \dfrac{1-|z|^2}{|1+z|^2} - i\dfrac{2\Im(z)}{|1+z|^2}$.

Notice that T maps $\mathbb{R} \setminus \{-1\}$ bijectively onto itself and $i\mathbb{R}$ bijectively onto $\mathbb{T} \setminus \{-1\}$. Also, $|z| < 1$ if and only if $\Re(w) > 0$ and $|w| < 1$ if and only if $\Re(z) > 0$.

We have seen that complex numbers can be viewed algebraically as numbers in a field and geometrically as points in a Euclidean plane. We can also view $z = x + iy$ as a two-dimensional vector in $\mathbb{R}^2$; here we are thinking of $\mathbb{C}$ as a real two-dimensional vector space. In fact $\mathbb{C}$ is a normed linear space where $|z|$ is the norm of the vector/number z. Moreover, this norm arises by way of the (Hermitian) inner product $< z, w >= z\bar{w}$. Notice that we recover the usual Euclidean 'dot product' of z and w by taking the real part of this; i.e., $z \cdot w = \Re(z\bar{w}) = xu + yv$ (when $z = x + iy$and $w = u + iv$). Thus, for example, z and w are orthogonal (as vectors in $\mathbb{R}^2$) precisely when $\Re(z\bar{w}) = 0$.

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From a topological viewpoint, $\mathbb{C}$ is a metric space where the distance function is given in the usual way by the underlying norm. Thus, $|z-w|$ is the distance between the complex numbers z and w; of course, this is just the ordinary Euclidean distance between the points z and w in the plane. In this regard we must mention the Triangle Inequalities:

$$||z|-|w|| \le |z \pm w| \le |z| + |w|.$$

In conclusion, we can view $\mathbb{C}$ in several different ways: a number field, a metric (or topological) space, a (two-dimensional) real vector space, or the Euclidean plane. Because of these many options, complex numbers have become an indispensable tool employed throughout pure and applied mathematics as well as the physical and engineering sciences.

1.B. **Polar Coordinates.** Here is an especially useful way to describe complex numbers.

We start by defining, for a real number θ, $\exp(i\theta) = e^{i\theta} := \cos\theta + i\sin\theta$; thus we obtain a map $\exp : \mathbb{R} \to \mathbb{T}$. See §2.C for motivation as to why we use this definition.

Recall the notion of polar coordinates used in analytic geometry. For non-zero complex numbers, $r = |z|$ gives the distance from the point z to the origin. The other 'coordinate' is the so-called polar angle. We say θ is a *value of the argument of* z, denoted $\theta = \arg(z)$, if $z = |z|e^{i\theta}$; in other words, if $\Re(z) = |z|\cos\theta$ and $\Im(z) = |z|\sin\theta$. Thus the polar coordinate notation $z = re^{i\theta}$ means that $r = |z|$ and θ is some value of $\arg(z)$; this notation is only used for $z \in \mathbb{C}_*$.

Clearly if θ is one of the possible values of the argument of z, then the set of *all* possible values is given by $\{\theta + 2n\pi : n \in \mathbb{Z}\}$. (The reader should note—at least mentally—the set of all arguments of, say, ± 1, $\pm i$, and $1+i$.) Given $z \in \mathbb{C}_*$, we single out the unique value θ of $\arg(z)$ which satisfies $-\pi < \theta \le \pi$ and call this the *principal value of the argument of* z, or briefly, the *principal value of* $\arg(z)$; we denote this by $\mathrm{Arg}(z)$. Thus we have a mapping $\mathrm{Arg} : \mathbb{C}_* \to (-\pi, \pi]$ with the property that $z = |z|\exp[\mathrm{Arg}(z)]$ for all $z \in \mathbb{C}_*$. Note that the set of all possible values of the argument of $z \in \mathbb{C}_*$ can now be described by $\{\mathrm{Arg}(z) + 2n\pi : n \in \mathbb{Z}\}$. Finally, we point out the obvious, but oh so important, fact that Arg is <u>not</u> continuous (as a mapping from $\mathbb{C}_*$ to $\mathbb{R}$).

Recalling the angle addition identities for the sine and cosine functions we readily deduce the following.

1.1. **Proposition.** *For all* $\alpha, \beta \in \mathbb{R}$, $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$.

We are now ready to take a geometric look at complex multiplication. Writing $z = re^{i\theta}$ and $a = |a|e^{i\alpha}$, say, we find that $az = r|a|e^{i(\theta+\alpha)}$. Thus a geometric description for the mapping $z \mapsto az$ is that the vector z is scaled by $|a|$ and then rotated by $\arg(a)$. We can also provide pictures for the complex arithmetic operations: addition, multiplication, inversion, square root. See my class notes!

As another application of Proposition 1.1 we note that $\arg(ab) = \arg(a) + \arg(b)$. However, this is only true as a <u>set</u> equality. (The reader should be sure to understand the precise meaning of this last statement!) For example, it is rarely true that $\mathrm{Arg}(ab) = \mathrm{Arg}(a) + \mathrm{Arg}(b)$.

Polar coordinates are especially useful for determining powers z^p and roots $z^{1/p}$ where $p \in \mathbb{Z}$ is an integer. First we remark that

$$\rho e^{i\varphi} = re^{i\theta} \iff \rho = r \quad \text{and} \quad \varphi = \theta + 2k\pi \quad \text{for some integer } k.$$

As an example, the four fourth roots of $4i$ (i.e., the four numbers z with the property that $z^4 = 4i$) are easily found to be $\pm\sqrt{2}e^{i\pi/8}, \pm\sqrt{2}e^{i5\pi/8}$. Similar reasoning provides verification of the following.

1.2. **Proposition.** *Let $n \in \mathbb{N}$. Then each $z \in \mathbb{C}_*$ has exactly n distinct n^{th} roots; that is, there are n distinct complex numbers w with $w^n = z$. In fact, these roots are given (in polar coordinates) precisely via*

$$w = \rho e^{i\varphi} \quad \text{where } \rho^n = |z| \text{ and } \varphi = [\operatorname{Arg}(z) + 2k\pi]/n \text{ for } k = 0, 1, \dots, n-1.$$

We emphasize that the notation $z^{1/n}$ is ambiguous in that it stands for any one of the n distinct roots of z. Typically which root is understood from context, but the reader is urged to use this notation with extreme caution! It is convenient to let

$$\sqrt[n]{z} := \rho \exp\left(\frac{i}{n}\operatorname{Arg}(z)\right) \quad \text{where } \rho^n = |z|$$

denote the *principal value of the n^{th} root of z*. Thus, for example, $\sqrt{z} := \sqrt[2]{z} = \rho e^{(i/2)\operatorname{Arg}(z)}$ (where $\rho^2 = |z|$) is the *principal value of the square root of z*; note that $z^{1/2}$ can be either $+\sqrt{z}$ or $-\sqrt{z}$.

For later use we now define the *oriented angle from z to w* (for $z, w \in \mathbb{C}_*$) by

$$\Theta(z, w) := \operatorname{Arg}(w/z).$$

Notice that w/z is what we must multiply z by to get w, so $\Theta(z, w)$ is the (principal value of the) angle we must rotate z by to get the direction given by w. As simple examples, $\Theta(1, 1+i) = \pi/4$ and $\Theta(i, 1) = -\pi/2$.

Here are some easy to check properties of this concept.

1.3. **Proposition.** *Let z and w be non-zero complex numbers. Then:*

(a) *$\theta = \Theta(z, w)$ is a signed angle in $(-\pi, \pi]$ and $|\theta|$ is the measure of the smaller of the two angles formed by the vectors z and w at the origin.*
(b) *$\Theta(z, w) = \pi$ if and only if $w = tz$ for some $t < 0$.*
(c) *When $\Theta(z, w) \neq \pi$, $\Theta(w, z) = -\Theta(z, w) = \Theta(\bar{z}, \bar{w})$.*
(d) *When $\Theta(z, w) = \pi$, $\Theta(w, z) = \pi = \Theta(\bar{z}, \bar{w})$.*
(e) *$\Theta(-z, w) = \Theta(z, w) - \pi = \Theta(z, -w)$.*
(f) *For all $c \in \mathbb{C}_*$, $\Theta(cz, cw) = \Theta(z, w)$.*
(g) *For all positive r and s, $\Theta(rz, sw) = \Theta(z, w)$.*

♣♣♣ Most of these follow by looking at $\operatorname{Arg}(\zeta)$. ♣♣♣

We conclude this subsection by discussing the notion of a branch of the argument function. Let A be a subset of $\mathbb{C}_*$. A function $\vartheta : A \to \mathbb{R}$ is termed *a branch of the argument function* (or simply a *branch of* $\arg(z)$) *in* A provided ϑ is <u>continuous</u> and for all $z \in A$, $z = |z| \exp[\vartheta(z)]$ (i.e., $\vartheta(z)$ is some value of $\arg(z)$). The reader should convince herself that there does <u>not</u> exist a (continuous) branch of the argument in $\mathbb{C}_*$. In particular, $\operatorname{Arg}(z)$ is not a branch of $\arg(z)$ (in $\mathbb{C}_*$). Your proof probably can be modified to demonstrate that if A is any set containing the unit circle $\mathbb{T}$, then there does not exists a (continuous) branch of the argument in A. We will see that this seemingly trivial fact is at the heart of much of complex analysis!

Once again: there does <u>not</u> exist a (continuous) branch of the argument in $\mathbb{C}_*$. To remedy this situation, we typically 'cut' the plane from the origin out to infinity and look for continuous branches of $\arg(z)$ in such regions. To provide some concrete examples of branches of

$\arg(z)$ in various sets, let us start with $\mathrm{Arg}(z)$ which is continuous when restricted to the set $\mathbb{C} \setminus (-\infty, 0]$ (or any subset thereof) and thus defines a branch of $\arg(z)$ there, the so-called principal branch. As a second example, the function ϑ given by

$$\vartheta(z) = \begin{cases} \mathrm{Arg}(z) & \text{when } \Im(z) \geq 0 \\ \mathrm{Arg}(z) + 2\pi & \text{when } \Im(z) < 0 \end{cases}$$

defines a (continuous) branch of the argument in $\mathbb{C} \setminus [0, +\infty)$. At this time the reader should construct a branch of the argument in $\mathbb{C} \setminus \{te^{i\alpha} : 0 \leq t < \infty\}$. We ask: Does there exist a (continuous) branch of $\arg(z)$ defined in $\mathbb{C} \setminus \{te^{it} : t \geq 0\}$?

We emphasize that whether or not a given set supports a branch of the argument is purely a question of continuity!

1.C. **Analytic Geometry.** What are the complex equations for lines and conics?
♣♣♣ This would also be a good place to discuss central and inscribed angles! ♣♣♣

Let us recall the 'standard' equations for lines and circles. Using our distance function it is easy to express disks and circles: we write

$$D(a; r) = \{z : |z - a| < r\},\ D[a; r] = \{z : |z - a| \leq r\},\ \text{and}\ C(a; r) = \{z : |z - a| = r\}$$

(respectively) for the *open disk*, *closed disk*, and *circle* (respectively) with *center* a and *radius* r. See below for an alternative description of circles.

There are two standard was to express an equation for a line. As in calculus, we can parametrize the line through the points a, b and obtain $z = z(t) = a + t(b - a)$ $(t \in \mathbb{R})$. Alternatively, if we know a normal vector, say ν, then the line through the point a with normal ν consists of points z such that the vector $z - a$ is orthogonal to ν. Recalling that $\Re(z\bar{w})$ gives the Euclidean dot product of z and w, we deduce that this line is given by $\Re[(z - a)\bar{\nu}] = 0$. As an added bonus, we see that this technique also provides a description for the half-plane determined by this line and ν; namely, $\{z : \Re[(z - a)\bar{\nu}] > 0\}$.

We can also easily express the equations for conic sections. For example, an ellipse is the locus of points z with the property that the sum of the distances from z to two given points, say a and b, is a constant. Thus $\{z : |z - a| + |z - b| = 2r\}$ is the ellipse with foci a, b and major axis r. The reader should write down similar expressions for parabolas and hyperbolas.

It is straightforward to verify that the equation

$$Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0\ , \quad \text{with } A, C \in \mathbb{R},\ B \in \mathbb{C} \text{ and } |B|^2 > AC$$

represents a line if and only if $A = 0$ and otherwise (i.e., when $A \neq 0$) a circle. ♣♣♣ Should let C be the complex number. ♣♣♣

A careful analysis of the complex equation $az + b\bar{z} + c = 0$, with $a, b, c \in \mathbb{C}$, is a wonderful exercise in using complex analysis. Of course this one equation corresponds to two real linear equations (obtained by taking the real and imaginary parts of the left hand side) involving two real variables x, y. From sophmore linear algebra we know that the solution possibilties are precisely: no solutions, a unique solution, or infinitely many solutions. It is a worthy exercise to determine this and explicitly describe the solutions using only complex analysis. [Answer: The trivial case, when $a = b = 0$ has either no solutions (when $c \neq 0$) or every z a solution (when $c = 0$). Assume one of a or b is non-zero. Then we get: a unique solution if and only if $|a| \neq |b|$, in which case $z = (b\bar{c} - \bar{a}c)/(|a|^2 - |b|^2)$; a line if and only if $|a| = |b| \neq 0$

and $b\bar{c} = \bar{a}c$, and the line is given by $\Re[(z + c/2a)a\bar{c}] = 0$, which is the line through the point $-c/2a$ with normal vector $\bar{a}c$; otherwise, there are no solutions.]

♣♣♣ could mention conics here—these preserved by cplx linear maps ♣♣♣

For later use we set up the following notation: we let $\mathcal{L}$ and $\mathcal{C}$ denote the collections of all lines and all circles (respectively) in $\mathbb{C}$. Thus

$$\mathcal{L} := \{L \subset \mathbb{C} | L \text{ is a straight line}\} \qquad \text{and} \qquad \mathcal{C} := \{C \subset \mathbb{C} | C \text{ is a Euclidean circle}\}.$$

2. Elementary Mappings-part I

Here we begin our study of function theory. There are various types of functions including: real-valued functions of a real variable $u = u(x)$, complex-valued functions of a real variable $z = z(t)$, real-valued functions of a complex variable $u = u(z)$, and complex valued functions of a complex variable $w = f(z)$. We are especially interested in visualizing the 'action' of such function via appropriate pictures. In particular, given a set A in the z-plane, what is $f(A)$ in the w-plane?

As in undergraduate calculus, we can (and will) study polynomials, rational functions, trignometric functions, algebraic functions, and transcendental functions (including the exponential, logarithm and trignometric functions). We start with the simplest possible functions.

2.A. **Complex Linear Maps.** What are these?

Well, what are real linear maps? The answer depends on the context. In a linear algebra class, a linear map (usually described as a linear transformation) between two vector spaces is simply a function which preserves the (linear) structure. On the other hand, in a freshman calculus class, the so-called linear functions are those of the form $y = ax + b$ whose graphs are straight lines. (In linear algebra, such a function is termed an affine map.) Here we shall adopt the latter point of view, although as a word of caution, we remark that some texts might use the former.

Thus, *complex linear maps* are complex-valued functions of a complex variable of the form $w = az + b$ where $a, b \in \mathbb{C}$ are constants and $z \in \mathbb{C}$ is a complex variable; typically we assume that $a \neq 0$. Special cases include:

a *translation by* $b \in \mathbb{C}$: $w = z + b$,
a *rotation about the origin*: $w = e^{i\theta} z$,
a *dilation wrt the origin*: $w = kz$ $(k > 0)$.

A geometric description of the map $z \mapsto w = az + b$ is that: z gets rotated by $\arg(a)$, then dilated by $|a|$, and then translated by b. We emphasize this: *every* complex linear map L can be factored as

$$L = T \circ S \circ R \quad \text{where } R \text{ is a rotation, } S \text{ is a scaling(dilation), and } T \text{ is a translation.}$$

As a simple example, look at $w = 2iz + (1 - i)$.

We can consider these as maps $\mathbb{R}^2 \xrightarrow{L} \mathbb{R}^2$ and from linear algebra we know their matix representations. For example, the dilation $w = kz$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and a rotation (about the origin) by the angle θ is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Again, in general a complex linear map does not give a linear transformation of $\mathbb{R}^2$ to itself but rather an affine transformation.

It is not difficult to confirm the following fundamental properties of complex linear maps.

2.1. **Proposition.** *A complex linear map $w = L(z) = az + b$ with $a \neq 0$ is a complex polynomial of degree one which is a bijection of $\mathbb{C}$ with inverse $z = L^{-1}(w) = (w - b)/a$ another complex linear map. (In fact, L is a self-homeomorphism of the plane.)*

We also note that if L is a pure rotation or pure dilation or pure translation, then so is L^{-1}.

It turns out that complex linear maps possess many important properties which also hold for more general maps. However, the proofs for the linear maps are nice geometric exercises, so we explain the basic ideas here.

A map $f : \mathbb{C} \to \mathbb{C}$ is called a *similarity* if the image of every triangle Δ is a triangle $\Delta' = f(\Delta)$ which is similar to Δ. A more useful definition is that f is a similarity if and only if there is a $t > 0$ such that for all $z, z' \in \mathbb{C}$, $|f(z) - f(z')| = t|z - z'|$. (Alternatively, f is a similarity if and only if for all distinct points $a, b, c \in \mathbb{C}$, $|a - b|/|a - c| = |f(a) - f(b)|/|f(a) - f(c)|$.) When $t = 1$ we see that f preserves distances; such a map is called an *isometry*, and it maps every triangle Δ to a triangle Δ' which is congruent to Δ.

It is straightfoward to demonstrate that complex linear maps are similarities.

2.2. **Proposition.** *A complex linear map $L(z) = az + b$ is a similarity transformation; when $|a| = 1$ it is an isometry.*

Proof. $|L(z) - L(z')| = |(az + b) - (az' + b)| = |a(z - z')| = |a||z - z'|$. □

In fact, complex linear maps are precisely the *orientation-preserving similarities* of the Euclidean plane.

Now we want to prove that complex linear maps transform lines and circles to lines and circles respectively. This is geometrically transparent for translations and rotations (isn't it?), but what about dilations? Here is an algebraic argument that should convince you. Let K be a line or a circle. Then K is given by some equation of the form

$$A|z|^2 + Bz + \bar{B}\bar{z} + C = 0\ , \quad \text{with } A, C \in \mathbb{R},\ B \in \mathbb{C} \text{ and } |B|^2 > AC$$

and K is a line if and only if $A = 0$ and otherwise (i.e., when $A \neq 0$) a circle. Now the mapping $w = kz$ transforms K into a set K' which is given by the equation

$$A'|w|^2 + B'w + \bar{B}'\bar{w} + C' = 0\ , \quad \text{with } A' = A/k^2, C' = C \in \mathbb{R} \text{ and } B' = B/k \in \mathbb{C}.$$

Since $|B'|^2 = |B|^2/k > AC/k^2 = A'C'$ and $A' = 0$ if and only if $A = 0$, we see that K' is a line or a circle just as K is.

The reader should convince herself that a similar argument works for rotations and translations.

Here is a geometric proof which works in general.

2.3. **Proposition.** *Let L be a complex linear map. If K is a line (or circle, respectively), then so is $L(K)$ (respectively).*

Proof. Assume that $w = L(z) = az + b$. First, suppose $K = C(z_0; r)$ is the circle centered at z_0 and of radius r. Let $w_0 = L(z_0)$. Suppose $w = L(z)$ for some $z \in K$. Recalling the proof of Proposition 2.2 we obtain

$$|w - w_0| = |L(z) - L(z_0)| = |a||z - z_0| = |a|r.$$

This means that $L(K) \subset K'$ where $K' = C(w_0; r')$ is the circle centered at w_0 of radius $r' = |a|r$. Exactly the same argument, applied to the inverse map L^{-1}, gives us $L^{-1}(K') \subset K$. Thus $L(K) = K'$.

Next, suppose K is a straight line. Select two points z_1 and z_2 so that K is the perpendicular bisector of the segment $[z_1, z_2]$; i.e., $K = \{z : |z - z_1| = |z - z_2|\}$. Let $w_i = L(z_i)$ for $i = 1, 2$ and define K' to be the perpendicular bisector of the segment $[w_1, w_2]$; so, $K' = \{w : |w - w_1| = |w - w_2|\}$. If $w = L(z)$ for some $z \in K$, then

$$|w - w_1| = |L(z) - L(z_1)| = |a||z - z_1| = |a||z - z_2| = |L(z) - L(z_2)| = |w - w_2|.$$

Thus $L(K) \subset K'$. Similarly, $L^{-1}(K') \subset K$, so $L(K) = K'$. □

Here are three noteworthy consequences of the above.

2.4**. Corollary.** *Let C be a circle in $\mathbb{C}$.*

(a) *For any points z_1, z_2, $L([z_1, z_2]) = [L(z_1), L(z_2)]$.*
(b) *If $[z_1, z_2]$ is a diameter of a circle C, then its image is a diameter of $L(C)$.*
(c) *If T is a line tangent to a circle C, then $L(T)$ is a line tangent to $L(C)$.*

In particular we notice from the above that if T is a line tangent to a circle C, and D is the line through the center of C and the point of interesction of C and T, then T and D are orthogonal and *so are their images* under any complex linear map. What about other angles?

We next verify the angle preservation property: If two lines (or two circles or a line and a circle) intersect, then so do their images and moreover the angles of intersection (for the original sets and their images) are exactly the same. This property just mentioned, angle preservation, is especially important. In fact, complex linear maps preserve oriented angles; mappings which possess this property are called *conformal*. The map $z \mapsto \bar{z}$ preserves angles (so it is an *isogonal* map), but does not preserve orientation and so is not conformal (but rather is *anti-conformal*).

The reader no doubt understands (from calculus or linear algebra) what is meant by the angle between two lines (or two circles or a line and a circle). As a warm-up for later purposes, when we study this in general for two intersecting paths, we now formalize this concept. Recall that for $z, w \in \mathbb{C}_*$, $\Theta(z, w) = \operatorname{Arg}(w/z)$ is the oriented angle from z to w.

Suppose that A and B are two lines which intersect at some point z_0. Pick direction vectors a, b for A, B (respectively); thus A, B are given by the parametric equations $\alpha(t) = z_0 + ta$, $\beta(t) = z_0 + tb$ (respectively) where $t \in \mathbb{R}$. We note that here we are actually considering A, B as *oriented* lines: $-a$ is also a direction vector for the line A and it gives the opposite orientation. The *oriented angle from A to B* is $\Theta(A, B) = \Theta(a, b) = \operatorname{Arg}(b/a)$. Note that changing the orientation of A (from that given by a to that given by $-a$) changes this angle by π; see Proposition 1.3(e). Also, $A = B$ as oriented lines if and only if the angle between them is zero; when this angle is π, then the lines are the same but they have opposite orientations.

It is now a simple matter to define the angle between two intersecting oriented circles or between an oriented circle and oriented line which meets it. Let K and K' each be an oriented line or circle (so, two lines or two circles or one of each) which intersect at some point. Let T, T' be oriented lines which are tangent to K, K' (respectively) at the point of intersection (with the orientations induced by the orientations of K, K'). Then the *oriented angle from K to K'* is $\Theta(K, K') = \Theta(T, T')$. Note that two circles (for example) are tangent at a point exactly when the angle between them is either 0 or π: which value we get depends both on whether the circles have opposite or identical orientation as well as whether they are internally tangent or externally tangent.

The reader is encouraged to not get too tangled up with the above formal definitions!

It is now rather straightforward to demonstrate that complex linear maps are conformal. Notice that if K is an oriented line or circle, then so is its image under any complex linear map.

2.5. **Proposition.** *Let L be a complex linear map. Suppose A and B are two oriented lines (or two oriented circles or an oriented line and an oriented circle) which intersect. Then so do their images $A' = L(A)$ and $B' = L(B)$, and moreover $\Theta(A', B') = \Theta(A, B)$.*

Proof. (Here is an outline for a formal proof.) First, the reader should convince herself that we need only consider the case when both A, B are oriented lines. Next, the assertion is immediate for translations (right?). Now look at the special case when A, B intersect at the origin and $L(0) = 0$. In this case $L(z) = cz$, so the lines A, B are simply rotated through the angle $\text{Arg}(c)$ and hence the oriented angle of intersection is unchanged.

Finally, consider the general case. Let $w_0 = L(z_0)$ where z_0 is the point common to A, B. The translations $T(z) = z - z_0$ and $S(w) = w - w_0$ preserve the oriented angles $\Theta(A, B)$ at z_0 and $\Theta(A', B')$ at w_0 (respectively). Put $F = S \circ L \circ T^{-1}$. Then F is a complex linear map with $F(0) = 0$. Since $T(A)$, $T(B)$ intersect at the origin, F preserves their oriented angle of intersection. It now follows that

$$\Theta(A', B') = \Theta\left(S^{-1}(F[T(A)]), S^{-1}(F[T(B)])\right) = \\ = \Theta\left(F[T(A)], F[T(B)]\right) = \Theta\left(T(A), T(B)\right) = \Theta(A, B).$$

□

Later we will examine functions of the form $w = (az + b)/(cz + d)$; these are the simplest possible rational functions and are called Möbius transformations. We will see that these maps possess many of the same properties enjoyed by the linear functions.

2.B. **Non-injective Maps.** Here we seek a geometric description of the polynomial map

$$z \mapsto w = p(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

Basic example: $w = z^2$; lotsa pictures including the images and preimages of straight lines; angles at the origin are doubled whereas away from the origin this is isogonal; lines,circles are not mapped to lines,circles; much more complicated than linear maps; inverses are $\pm\sqrt{z}$.

Another example: $w = z + 1/z$

2.C. **The Complex Exponential Function.** What should e^z be for $z \in \mathbb{C}$?

We assume knowledge about e^x for $x \in \mathbb{R}$. For $z = x + iy$ we define

$$\exp(z) = e^z := e^x e^{iy} = e^x \cos y + i\, e^x \sin y.$$

Recall from the beginning of §1.B that for a real number θ we have defined $e^{i\theta} := \cos\theta + i \sin\theta$. To understand why we use this definition of $e^{i\theta}$, we first note that for $n \in \mathbb{N}$, i^n is either ± 1

(when n is even) or $\pm i$ (when n is odd). In particular, if $n = 2k$, then $i^n = (-1)^k$ and if $n = 2k+1$, then $i^n = (-1)^k i$. Now we use property (1) below and (formally) manipulate power series as follows:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} = \cos(\theta) + i\sin(\theta).$$

Properties of e^z

(1) $e^z = \sum_0^\infty z^n/n!$ and this converges absolutely for all $z \in \mathbb{C}$.
(2) $|e^z| = e^{\Re(z)}$ and $\arg(e^z) = \Im(z) + 2k\pi$ for $k \in \mathbb{Z}$.
(3) $e^z e^w = e^{z+w}$, $e^{-z} = 1/e^z$, and for all $n \in \mathbb{N} : (e^z)^n = e^{nz}$.
(4) $e^w = e^z$ if and only if $w = z + 2k\pi i$ for some $k \in \mathbb{Z}$.

Remarks

(1) $|e^z| = e^{\Re(z)} > 0$ implies in particular that $e^z \neq 0$ for all $z \in \mathbb{C}$. We shall see that every non-zero complex number w can be written as e^z; in fact this can be done for infinitely many z. Thus every point $w \in \mathbb{C}$, except $w = 0$, is covered infinitely often by the mapping $w = e^z$.
(2) $\lim_{z\to\infty} e^z$ does not exist. In fact, by choosing an appropriate sequence of points $z_n \to \infty$, you can get any value you want for $\lim_{n\to\infty} e^{z_n}$. Another, more sophisticated, way of expressing this is that in every neighborhood of infinity, $w = e^z$ comes arbitrarily close to every complex number infinitely often.

Mapping properties of $w = e^z$: look at the images of horizontal and vertical lines; the image of a rectangle $\{x + iy : a \le x \le b, c \le y \le d\}$ with $d - c \le 2\pi$ is the 'circular rectangle' $\{re^{i\theta} : e^a \le r \le e^b, c \le \theta \le d\}$; what is the image of a line $\{x + imx : x \in \mathbb{R}\}$?

2.D. **Trignometric and Hyperbolic Functions.** These are defined as follows:

$$\cos(z) = (e^{iz} + e^{-iz})/2, \quad \sin(z) = (e^{iz} - e^{-iz})/2i$$

and

$$\cosh(z) = (e^z + e^{-z})/2, \quad \sinh(z) = (e^z - e^{-z})/2.$$

2.E. **The Complex Logarithm Function.** What should $\log(z)$ be for $z \in \mathbb{C}$?

Recall that for real x, the functions $x \mapsto e^x$ and $x \mapsto \ln x$ are inverses of each other; we assume knowledge concerning the natural logarithm function $\ln x$ for $x \in \mathbb{R}$. We want to define an inverse for the complex exponential function, but of course this does not make sense because $z \mapsto e^z$ is not one-to-one (in fact, is far from it). In spite of this, let us try to do it anyway and see where this leads. We begin with a fixed $z \in \mathbb{C}$ and try to solve the equation $e^w = z$ for w. Obviously this cannot have a solution unless $z \neq 0$, so we assume this. Then we can express z in polar coordinates as $z = re^{i\theta}$ where θ is some value of $\arg(z)$. Now write $w = u + iv$. Then we have

$$re^{i\theta} = z = e^w = e^u e^{iv}$$

which leads to $r = e^u$ and $\theta = v + 2k\pi$ for some $k \in \mathbb{Z}$. From this we conclude that

$$u = \ln(r) = \ln|z| \quad \text{and} \quad v = \theta + 2k\pi \quad \text{for some } k \in \mathbb{Z}.$$

Notice that v above is equal to some value of $\arg(z)$. We record our observations as follows.

2.6. **Lemma.** *For $z \in \mathbb{C}_*$, $e^w = z \iff w = \ln|z| + i\arg(z)$.*

The reader should be certain (s)he understands the precise meaning of the above! It is not hard to see that when w has such a form, then $e^w = z$; the necessity of this is what we verified above. Notice that this Lemma says that $\Re(w) = \ln|z|$ for any w with $e^w = z$; note that $z \mapsto \ln|z|$ is a well-defined (single-valued) real-valued function of a complex variable. On the other hand, all we know about $\Im(w)$ is that it must be some value of $\arg(z)$.

Given $z \in \mathbb{C}_*$, we say that $w \in \mathbb{C}$ is a *value of the logarithm of z* (or more simply, a *value of* $\log(z)$) if $e^w = z$. Lemma 2.6 tells us precisely what these values are. As an example, the *principal value of* $\log(z)$ is

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z).$$

We cannot use the 'formula' for w provided by Lemma 2.6 to define a function $\log(z)$ because it does not provide a single-value but rather many possible values. To get a true (i.e. single-valued) function we must first choose a branch of the argument. Recall that any continuous real-valued function $\vartheta(z)$ satisfying $z = |z|e^{i\vartheta(z)}$ is termed a branch of the argument of z. Here, of course, the domain of ϑ must be a subset of $\mathbb{C}_*$, and this just means that $\vartheta(z)$ is one of the (many) values for $\arg(z)$.

Next, by a *branch of the logarithm function* we mean a (single-valued) complex-valued function $\lambda(z)$ with the property that $e^{\lambda(z)} = z$ for all z. Thus $\lambda(z)$ is a value of $\log(z)$ for each z and so λ serves as an inverse for the exponential function; hence the domain of λ must be a subset of $\mathbb{C}_*$. Again, we also want λ to be continuous, and this is impossible to do in all of $\mathbb{C}_*$, so usually we restrict our attention to the regions described at the end of §1.B (where we took a branch cut from the origin out to infinity).

To make a formal definition, a continuous function $\lambda : A \to \mathbb{C}$ (with $A \subset \mathbb{C}_*$) is a *branch of the logarithm function in A* (or more briefly, a *branch of* $\log(z)$ *in A*) provided $\exp(\lambda(z)) = z$ for all $z \in A$. Note: Any such λ serves as an inverse for the exponential function in A.

Let us emphasize the point that whenever we speak of a branch of the logarithm, there is always an associated domain-set. In fact, whether or not there exists a (continuous) branch of the logarithm in a given set A depends in a very strong way on the topology of A. Moreover, this problem (of determining when there exists a branch of $\log(z)$) is at the very heart of complex analysis.

As an example, the *principal branch of the logarithm* is defined for $z \in \mathbb{C} \setminus (-\infty, 0]$ via

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z).$$

As one might expect, there is a close tie between branches of the argument and branches of the logarithm. In fact we have the following result, whose proof mimics the argument given for Lemma 2.6.

2.7. **Proposition.** *A function $\lambda(z)$ is a branch of the logarithm if and only if there exists a branch of the argument, say $\vartheta(z)$, such that for all z, $\lambda(z) = \ln|z| + i\vartheta(z)$.*

We also have the following information about continuous branches of the logarithm.

2.8. **Theorem.** *Let λ_1 and λ_2 be two branches of the logarithm defined (and continuous) in the same open connected set Ω. Then there is a fixed $k \in \mathbb{Z}$ such that for all $z \in \Omega$, $\lambda_1(z) = \lambda_2(z) + 2k\pi i$.*

Proof. □

Note that for any branch $\lambda(z)$ of the logarithm, defined in some set A, we always have $e^{\lambda(z)} = z$ for all $z \in A$. However, we may not have $\lambda(e^z) = z$. Indeed, even for $z = x > 0$ we need not have $\log(x) = \ln(x)$. For example...

2.F. **Complex Powers and Roots.** For $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ we define

$$a^b := e^{\log a}.$$

Some special cases of this include:

(1) If $b \in \mathbb{Z}$, then a^b is a single value.
(2) If $b = p/q \in \mathbb{Q}$ is rational with p, q having no common factors, then there are q distinct complex values of a^b and these are symmetrically located around the circle $|w| = |a|^b$.
(3) If $b \in \mathbb{R} \setminus \mathbb{Q}$ is irrational, then a^b consists of infinitely many complex values located around the circle $|w| = |a|^b$.

Note however that the 'law of exponents' $a^b a^c = a^{b+c}$ fails to hold, even when this is considered as a set equality. For example...

2.G. **Inverse Functions.** The branches of the inverse of $y = x^2$ are $\pm\sqrt{x}$.

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