UNIVERSITY OF Cincinnati

Department of Mathematical Sciences 839 Old Chemistry Building
PO Box 210025 Phone (513) 556-4075
Cincinnati OH 45221-0025 Fax (513) 556-3417

COMPLEX ANALYSIS NOTES
WINTER QUARTER 2010

DAVID A HERRON

Contents

Date: January 15, 2010.

1. Complex Integration Theory

1.A. **Groupoid Properties of Paths.** Recall that a *path in* Ω is a continuous map $\mathbb{R} \supset [a,b] \xrightarrow{\gamma} \Omega$ and its *trajectory* (i.e., its image:-) is the set $|\gamma| := \gamma([a,b])$. Note that $|\gamma|$ is a compact connected subset of Ω; thus, e.g., $\operatorname{dist}(|\gamma|, \partial\Omega) > 0$. We think of γ as going from its *initial point* $\gamma(a)$ to its *terminal point* $\gamma(b)$. When $\gamma(a) = \gamma(b)$, we call γ a *loop*. (The term *closed path* is also used, but I think that this can be misleading.)

Please see Palka's book (and my class notes too) for a more thorough discussion of paths.

1.B. **Real Path Integrals.** Here we recall a few things from Calculus.

We call $p\,dx + q\,dy$ an *exact differential* if there is a function F satisfying $dF = p\,dx + q\,dy$; of course this means that

$$\frac{\partial F}{\partial x} = p \qquad \text{and} \qquad \frac{\partial F}{\partial y} = q\,.$$

Alternatively, we could ask that the vector field (p,q) be a conservative or gradient field, meaning that there exists some F with $\nabla F = (p,q)$. Here p,q,F should all have the same domain of definition, and can be either real or complex valued. Of course F should possess the indicated partial derivatives, and when p,q are continuous, F must be $\mathcal{C}^1$.

Now suppose that $p\,dx + q\,dy$ is exact in Ω, say equal to dF. Let's look at the integral of this along a path $\gamma : [a,b] \to \Omega$. We see that

$$\int_\gamma p\,dx + q\,dy = \int_a^b \frac{d}{dt}\,[F \circ \gamma]\,dt = F(\gamma(b)) - F(\gamma(a))\,.$$

In particular we see that the integral of an exact differential depends only on the endpoints of the path and not on the actual path we take from one endpoint to the other.

The converse is also true, and we record this as the following fundamental result regarding path integrals.

1.1. **Theorem** (FTRPI). *Let p and q be continuous in some domain Ω. Then the following are equivalent.*

(a) *$p\,dx + q\,dy$ is an exact differential in Ω.*
(b) *For any piecewise smooth path γ in Ω, the value of $\int_\gamma p\,dx + q\,dy$ depends only on the endpoints of γ.*
(c) *For any piecewise smooth closed loop Γ in Ω, $\int_\Gamma p\,dx + q\,dy = 0$.*

Proof. We have already indicated why (a) implies (b). We leave confirmation of (b) $\Longleftrightarrow$ (c) to the reader. Here we verify that (b) implies (a). So, assume (b) holds. We define an $F \in \mathcal{C}^1(\Omega)$ with the property that $dF = p\,dx + q\,dy$.

To this end, let z_0 be a fixed point in Ω. For each $z \in \Omega$, let γ_z denote some piecewise smooth path in Ω that joins z_0 to z. Then, for each $z \in \Omega$, we define

$$F(z) := \int_{\gamma_z} p\,dx + q\,dy\,.$$

Note that the hypotheses in (b) permit us to use any piecewise smooth path γ_z whatsoever, that joins z_0 to z in Ω. It remains to corroborate that

$$\forall\, z \in \Omega\,, \quad F_x(z) = p(z) \;\text{ and }\; F_y(z) = q(z)\,.$$

We give the details for the first of these; a proof of the second assertion is similar.

Let z_1 be a fixed point in Ω and let $\gamma_1 := \gamma_{z_1}$ be any fixed piecewise smooth path in Ω that joins z_0 to z_1. Since Ω is open, there is an $r > 0$ so that $D(z_1; r) \subset \Omega$. Then for each $h \in \mathbb{R}$ with $|h| < r$, the path

$$\gamma = \gamma_{z_1+h} := \gamma_1 + [z_1, z_1 + h]$$

is a piecewise smooth path in Ω that joins z_0 to $z_1 + h$. (Of course this path is just the concatenation of γ_1 with the line segment from z_1 to $z_1 + h$.) According to our definition of F,

$$\begin{aligned} F(z_1 + h) &= \int_\gamma p\,dx + q\,dy = \int_{\gamma_1+[z_1,z_1+h]} p\,dx + q\,dy \\ &= \int_{\gamma_1} (p\,dx + q\,dy) + \int_{[z_1,z_1+h]} (p\,dx + q\,dy) \\ &= F(z_1) + \int_{[z_1,z_1+h]} p\,dx + q\,dy \end{aligned}$$

and thus—recalling that $[0, 1] \ni t \mapsto z_1 + th$ parametrizes $[z_1, z_1 + h]$—we obtain

$$F(z_1 + h) - F(z_1) = \int_{[z_1,z_1+h]} (p\,dx + q\,dy) = \int_0^1 p(z_1 + th)\,h\,dt$$

and so

$$\frac{F(z_1 + h) - F(z_1)}{h} = \int_0^1 p(z_1 + th)\,dt\,.$$

We could now appeal to the Fundamental Theorem of Calculus, but it is easy enough to finish the proof. Let $\varepsilon > 0$ be given. Since p is continuous, we may select a $\delta \in (0, r)$ so that

$$\forall\, z \in D(z_1; \delta) \subset \Omega\,, \quad |p(z) - p(z_1)| < \varepsilon\,.$$

Then for each $h \in \mathbb{R}$ with $0 < |h| < \delta$, $[z_1, z_1 + h] \subset D(z_1; \delta)$, so

$$\frac{F(z_1 + h) - F(z_1)}{h} - p(z_1) = \int_0^1 p(z_1 + th)\,dt - p(z_1) = \int_0^1 [p(z_1 + th) - p(z_1)]\,dt$$

and thus

$$\left|\frac{F(z_1 + h) - F(z_1)}{h} - p(z_1)\right| \le \int_0^1 |p(z_1 + th) - p(z_1)|\;dt \le \int_0^1 \varepsilon\,dt = \varepsilon\,.$$

It now follows that

$$\lim_{h\to 0} \frac{F(z_1 + h) - F(z_1)}{h} = p(z_1) \quad \text{so} \quad \frac{\partial F}{\partial x}(z_1) = p(z_1)\,.$$

Since z_1 was an arbitrary point of Ω, $F_x = p$ everywhere in Ω. □

We also have path integrals with respect to arclength; we note that ds is not a differential!

1.C. **Complex Path Integrals.** Let $\mathbb{R} \supset [a,b] \xrightarrow{\gamma} \mathbb{C}$ be a piecewise smooth path. Suppose $|\gamma| \xrightarrow{f} \mathbb{C}$ is continuous. Then the *integral of f along γ* is

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\dot\gamma(t)\,dt\,.$$

The following is a simple, but oh-so-important, fundamental example.

1.2. *Example* (FX). For any $a \in \mathbb{C}$ and each $r > 0$,

$$\int_{C(a;r)} \frac{dz}{z-a} = 2\pi i\,.$$

Complex path integrals enjoy the usual linearity properties (of course here with respect to complex coefficients).

Recall that F is an *holomorphic anti-derivative* of f if F is holomorphic and $F' = f$. As simple, but important, examples we note that e^z and all complex polynomials have *entire* holomorphic anti-derivatives.

Here is a simple but surprisingly useful fact.

1.3. **Proposition.** *Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ be continuous. Then $f\,dz$ is an exact differential if and only if f has a holomorphic anti-derivative in Ω.*

Proof. If there exists a $\mathcal{C}^1$ function $\mathbb{C} \supset \Omega \xrightarrow{F} \mathbb{C}$ with

$$f\,dx + if\,dy = f\,dz = dF = F_x\,dx + F_y\,dy\,,$$

then $F_x = f = -iF_y$ so F is holomorphic in Ω. The converse is easy to check. □

As a corollary we obtain the following complex version of the fundamental theorem for path integrals.

1.4. **Theorem** (FTPI). *Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ be continuous. Then the following are equivalent.*

(a) *$f\,dz$ is an exact differential in Ω.*

(b) *f has a single-valued holomorphic anti-derivative in Ω.*

(c) *For any piecewise smooth path γ in Ω, the value of $\int_\gamma f(z)\,dz$ depends only on the endpoints of γ.*

(d) *For any piecewise smooth closed loop Γ in Ω, $\int_\Gamma f(z)\,dz = 0$.*

1.5. **Corollary.** *It is not possible to define a single-valued holomorphic branch of* $\log z$ *in* $\mathbb{C}_*$.

Here is a nice application of the fundamental theorem for path integrals; we will use this in our proof of Cauchy's Integral Formula for a circle.

1.6. **Lemma.**

$$\int_{C(a;r)} \frac{d\zeta}{\zeta - z} = \begin{cases} 2\pi i & \text{for } |z-a| < r\,, \\ 0 & \text{for } |z-a| > r,\,. \end{cases}$$

Proof. In both situations there is an appropriate single-valued holomorphic branch of a certain logarithm in the necessary domain. □

2. Cauchy Theory–Part I

Here we present elementary versions of Cauchy's Theorems and Integral Formulas. The latter provide a key tool for proving many basic results in complex function theory.

2.A. **Cauchy's Theorems.**

We begin this subsection with one of the most important results from multivariable calculus, Green's Theorem. We only require this in its simplest form, in which case the proof is immediate.

2.1. **Theorem** (GTR)**.** *Let p, q be $\mathcal{C}^1$ in an open set $\Omega \subset \mathbb{C}$. Suppose a closed rectangle R, with horizontal and vertical edges, lies inside Ω. Then*

$$\int_{\partial R} p\,dx + q\,dy = \iint_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy\,.$$

Now we start our investigation into Cauchy's Theorems: there are many 'flavors' of these each of which says, roughly,

$$\text{for any holomorphic } f \text{ and pcws smooth loop } \Gamma, \quad \int_\Gamma f(z)\,dz = 0\,.$$

Of course, since

$$\int_{\mathbb{T}} \frac{dz}{z} = 2\pi i$$

we see that there must some additional hypotheses! In fact there will **always** be some kind of topological hypothesis, either a restriction concerning the type of path Γ, or something about the region/domain in question (that is, the location of Γ). We draw attention to which type of hypothesis is in play by employing the preposition **for** in the former situation and **in** for the latter. The reader is encouraged to pay attention!

We begin by considering Cauchy's Theorem for a Rectangle (CTR). We present two proofs of this result, first explaining the classical proof that utilizes Green's Theorem, and then giving Goursat's proof via a bisection method.

2.2. **Theorem** (CTR)**.** *Let $f \in \mathcal{H}(\Omega)$. Suppose that a closed rectangle R (its interior and boundary) lies inside Ω. Then*

$$\int_{\partial R} f(z)\,dz = 0\,.$$

It will be especially useful to know the following modified version of Cauchy's Theorem for a Rectangle.

2.3. **Theorem** (CTR′)**.** *Fix $\zeta \in \Omega \subset \mathbb{C}$. Let $f \in \mathcal{H}(\Omega \setminus \{\zeta\})$. Assume that*

$$\lim_{z \to \zeta} (z - \zeta) f(z) = 0\,.$$

Then for each closed rectangle $R \subset \Omega$ with $\zeta \notin \partial R$,

$$\int_{\partial R} f(z)\,dz = 0\,.$$

Note that, by induction, we can actually have a finite number of exceptional points ζ.

With the aid of Cauchy's Theorem for a Rectangle, we can now prove Cauchy's Theorem in a Disk. (Again, note the difference between the uses of the prepositions **for** and **in**!) Recall too the Fundamental Theorem for Path Integrals.

2.4. **Theorem** (CTD)**.** *Let f be holomorphic in a disk $D := D(a; r)$. Then for any piecewise smooth loop Γ in D,*

$$\int_\Gamma f(z)\, dz = 0\,.$$

As with CTR, we require the following modified version of CTD.

2.5. **Theorem** (CTD′)**.** *Let $\zeta_1, \ldots, \zeta_n$ be points in a disk $D := D(a; r)$. Suppose that f is holomorphic in $\Omega := D \setminus \{\zeta_1, \ldots, \zeta_n\}$ and that*

$$\text{for each exceptional point } \zeta_k\,, \qquad \lim_{z \to \zeta_k} (z - \zeta_k) f(z) = 0\,.$$

Then for any piecewise smooth loop Γ in $\Omega \setminus \{\zeta_1, \ldots, \zeta_n\}$

$$\int_\Gamma f(z)\, dz = 0\,.$$

2.B. **Cauchy's Integral Formulas.** The immediate consequences of CIF include virtually every basic result in Complex Function Theory. To name a few we mention: connection with power series (holomorphic functions are real analytic), CIF for derivatives, holomorphic functions have holomorphic derivatives, Morera's Theorem, Liouville's Theorem, Fundamental Theorem of Algebra, Cauchy Estimates, Casorati-Weierstrass Theorem, Mean Value Property, Maximum Principle, Schwarz' Lemma, Riemann's Extension Theorem, Argument Principle, Residue Theorem.

Here is Cauchy's Integral Formula for a Circle.

2.6. **Theorem** (CIFC)**.** *Let $f \in \mathcal{H}(\Omega)$. Suppose that Ω contains the closed disk $D[a; r]$. Then for each $z \in D(a; r)$*

$$f(z) = \frac{1}{2\pi i} \int_{C(a;r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

As always we can have a finite number of exceptional points, although in this case we must of course require that none of these lie on the path of integration.

The proof of CIF is an easy consequence of CTD′ and Lemma 1.6.

Our first application of CIF is Cauchy's Integral Formulas for derivatives (again just for a circle).

2.7. **Theorem.** *Let $f \in \mathcal{H}(\Omega)$. Suppose that Ω contains the closed disk $D[a; r]$. Then f', f'', etc. are all holomorphic in Ω and for each $z \in D(a; r)$*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C(a;r)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta\,.$$

This is actually a consequence of the more general result, Lemma 3 on p.121 in Ahlfors.

Department of Mathematics, University of Cincinnati, OH 45221
E-mail address: David.Herron@math.UC.edu