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COMPLEX ANALYSIS HOMEWORK PROBLEMS
SPRING QUARTER 2010

Please provide plenty of details! Pix are definitely kewl ($\ddot{\smile}$).

(1) Please be sure to read Ahlfors and look at (and work) the suggested problems; check the web page on a regular basis. For now, start reading §4 of Chapter 4.

(2) You can also look at Palka's book: see his §§5,6 of Chapter 5.

(3) Let Ω be a bounded plane domain containing the origin. Suppose $\Omega \xrightarrow{f} \Omega$ is holomorphic with $f(0) = 0$ and $f'(0) = 1$. Prove that for all $z \in \Omega$, $f(z) = z$. (Hints: It suffices to show that $f(z) = z$ for all $z \in \Delta$ where $\Delta := D(0; r)$ and $0 < r < \text{dist}(0, \partial\Omega)$. Look at the Maclaurin series for f, say $f(z) = z + a_m z^m + \dots$. Use Cauchy estimates to find a bound for $|a_m|$. Assume that $a_m \neq 0$ and examine the Maclaurin series for the k-fold composition $f \circ f \circ \dots \circ f$ and consider 'what happens' as $k \to \infty$.)

(4) Read pp.1-136 in Ahlfors and do problems 1-4 on p.133.

(5) Read pp.152-154 in Ahlfors and do problems 1-3 on p.154.

(6) Determine the number of solutions to $z^4 - 5z^2 + 3 = e^{-z}$ in the closed right half-plane $\bar{\mathbb{H}} = \{\mathfrak{Re}(z) \geq 0\}$.

(7) Show that $f(z) = z^4 - 3z^2 + 3$ has exactly one zero in the open first quadrant $Q = \{\mathfrak{Re}(z) > 0, \mathfrak{Im}(z) > 0\}$. (Hint: Use the Argument Principle.)

(8) Prove the following generalization of Rouché's Theorem: Let $f, g \in \mathcal{H}(\Omega)$. Let Γ be a positively oriented piecewise smooth Jordan loop with the closure of $\mathsf{int}(\Gamma)$ contained in Ω. Suppose that for all $z \in |\Gamma|$,

$$|f(z) - g(z)| < |f(z)| + |g(z)|\,.$$

Then f and g have exactly the same number of zeroes in $\mathsf{int}(\Gamma)$, counted according to multiplicity.

(9) Use Rouché's Theorem to prove Hurwitz' Theorem.

(10) Let $(f_n)_1^\infty$ be a sequence of functions that are holomorphic and univalent in a domain Ω. Suppose that $(f_n)_1^\infty$ converges normally in Ω to some f. Prove that either f is a constant map or f is univalent in Ω.

(11) Let $h(z) := z(z^2 - 1)$. For each of the following sets A, determine whether or not there exists a branch of the cube root of h in $\mathbb{C} \setminus A$.
(a) $A = (-\infty, -1] \cup \{iy : y \geq 0\} \cup [1, +\infty)$; (b) $A = [-1, 1]$;
(c) $A = \{iy : y \geq 0\} \cup \{e^{it} : -\pi \leq t \leq 0\}$; (d) $A = [-1, 0] \cup [1, \infty)$.

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(12) Suppose that a map f is defined and holomorphic in some region $|z| > R$. As discussed on page 129 in Ahlfors, in this situation we say that f has an isolated singularity at ∞. The nature of this singularity is determined by looking at the map g defined by $g(z) := f(1/z)$: f has a removable singularity (or pole or essential singularity) at ∞ if and only if g has a removable singularity (or pole or essential singularity, respectively) at 0. When this singularity is removable, we say that f is holomorphic at ∞; be careful—we have not proven any theorems about such a map (only theorems about the auxiliary map g).

Prove that f has a non-essential singularity at ∞ if and only if there exists a point $a \in \hat{\mathbb{C}}$ with the property that

$$\lim_{z\to\infty} \chi(f(z), a) = 0$$

where χ is the chordal distance in $\hat{\mathbb{C}}$.

(13) Let f be holomorphic and non-constant in a domain Ω. Prove that whenever B is a discrete set in $\mathbb{C}$, $A := f^{-1}(B)$ is a discrete set in Ω. (If you argue by way of contradiction, be very careful that you actually arrive at a contradiction!)

(14) Suppose $f \in \mathcal{M}(\Omega_f)$ and $g \in \mathcal{M}(\Omega_g)$. Determine conditions (on f, g and/or on Ω_f, Ω_g) that describe when fg, $f + g$, f/g and $f \circ g$ are meromorphic.

Suppose that f is meromorphic in a neighborhood U of some $a \in \hat{\mathbb{C}}$. We define the *multiplicity* $m(f; a)$ *of* f *at* a as follows:

- If f is constant in U, then we set $m(f; a) := \infty$.
- If f has a pole of multiplicity m at a, then we set $m(f; a) := m$.
- If f is holomorphic and non-constant in U, then $m(f; a)$ is the multiplicity of the zero of the function $z \mapsto f(z) - f(a)$ at a.

Now let $f \in \mathcal{M}(\Omega)$. The *counting function* $\hat{\mathbb{C}} \xrightarrow{\nu_f} \mathbb{Z}_+$ *for* f is defined by

$$\nu_f(c) := \sum_{a\in f^{-1}(c)\cap\Omega} m(f; a)\,.$$

Thus for each $c \in \hat{\mathbb{C}}$, $\nu_f(c)$ is the number of time that f takes on the value c, counted according to multiplicity.

(15) Prove that the counting function for a polynomial is a constant function.

(16) Let f be a rational function. Find the counting function ν_f.

(17) (a) Let A, B be disjoint closed plane sets with B bounded. Construct a cycle Σ in $\mathbb{C} \setminus (A \cup B)$ with the property that

$$n(\Sigma; z) = \begin{cases} 0 & \text{for } z \in A\,, \\ 1 & \text{for } z \in B\,. \end{cases}$$

(b) Can you 'make' Σ a loop? A Jordan loop?

(c) Let A, B_1, B_2 be disjoint closed plane sets with B_1, B_2 bounded. Construct cycles Σ_1, Σ_2 in $\mathbb{C} \setminus (A \cup B_1 \cup B_2)$ with the property that for $1 \le i, j \le 2$ and $i \ne j$

$$n(\Sigma_j; z) = \begin{cases} 0 & \text{for } z \in A \cup B_i\,, \\ 1 & \text{for } z \in B_j\,. \end{cases}$$

(18) Construct a simply connected domain whose complement is the disjoint union of an infinite number of closed connected sets.

(19) Suppose a domain Ω has the property that each point z in $\mathbb{C}\setminus\Omega$ lies in some unbounded connected subset A_z of $\mathbb{C} \setminus \Omega$. Prove that Ω is simply connected.

(20) Prove that a domain Ω is simply connected if and only if each $f \in \mathcal{H}(\Omega)$ possess an anti-derivative that is holomorphic in Ω.

(21) Let Ω be any simply connected region that does not contain the origin. Prove that in Ω there exist holomorphic branches of: the logarithm function, each p^{th}-root function ($p \in \mathbb{N}$), and even a branch of z^z.

(22) Suppose Ω is simply connected and $f \in \mathcal{H}(\Omega)$. Let Γ be a piecewise smooth loop in Ω. Prove that for each $b \in \mathbb{C} \setminus f(\Omega)$, $n(f \circ \Gamma; b) = 0$. Does this imply that the domain $f(\Omega)$ is simply connected?

(23) Prove that a domain Ω is simply connected if and only if for every $f \in \mathcal{H}(\Omega)$ that is zero free in Ω there is a holomorphic branch of the square root of f in Ω. Is this fact true if 'square' root is replaced by 'p^{th}-root' for some integer $p \geq 3$?

(24) Let Ω be a domain with the property that ± 1 lie in the same component of the complement of Ω. Prove that there exists an $h \in \mathcal{H}(\Omega)$ that satisfies: for all $z \in \Omega$, $[h(z)]^2 = 1 - z^2$. Determine all possible values of

$$\int_\Gamma \frac{dz}{h(z)} \quad \text{for } \Gamma \text{ a piecewise smooth loop in } \Omega\,.$$

(25) Use our Factor Theorem for Poles to derive our formulas for the residue at a pole.

(26) Read pp.154-161 in Ahlfors and do problems 1, 3, 4 on p.161.

(27) Read pp.175-186 in Ahlfors and do problems:
2,4,5 on pp.178-179 ; 1,3,5 on p.184; 2,3 on p.186.

(28) Verify that each of the following converges and determine their values.

$$\int_1^\infty \frac{dx}{1+x^2}\,, \qquad \int_0^\infty \frac{x\sin(x)}{x^2+a^2}\,dx\,, \qquad \int_0^\infty \frac{x^\alpha}{(1+x^2)^2}\,dx \ \text{ (where } -1 < \alpha < 3)$$

(29) Let f be holomorphic in Ω except for isolated singularities in some set $S \subset \Omega$. Prove that f has a holomorphic anti-derivative in Ω if and only if for each $a \in S$, $\mathsf{Res}(f; a) = 0$.

(30) Let $f \in \mathcal{H}(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$ and non-constant. Suppose that for all $z \in \mathbb{T}$, $|f(z)| \leq 1$.
(a) Prove that f has at least one fixed point in $\bar{\mathbb{D}}$.
(a) Suppose for all $z \in \mathbb{T}$, $f(z) \neq z$. Prove that f has exactly one fixed point in $\bar{\mathbb{D}}$.

(31) Let $\mathcal{F}$ be a family of maps $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ that are continuous in the open set Ω. Prove that if $\mathcal{F}$ is normal in Ω, then it is both equicontinuous in Ω and locally uniformly bounded in Ω. (I suggest proving the contrapositive or each of these. Thus, for example, suppose that $\mathcal{F}$ is not locally uniformly bounded and verify that there exist a sequence (f_n) in $\mathcal{F}$ that fails to converge normally in Ω.)

(32) Let $\mathcal{F} \subset \mathcal{H}(\Omega)$. Put $\mathcal{F}' := \{f' \mid f \in \mathcal{F}\}$.
(a) Prove that if $\mathcal{F}$ is locally uniformly bounded in Ω, then so is $\mathcal{F}'$.
(b) Prove that if $\mathcal{F}$ is normal in Ω, then so is $\mathcal{F}'$.

(33) Recall that $\mathcal{H}(\Omega, \Omega') := \{f \in \mathcal{H}(\Omega) \mid f(\Omega) \subset \Omega'\}$. Suppose that Ω' is a non-dense subset of $\mathbb{C}$. What can you say about any sequence $(f_n)_1^\infty$ in $\mathcal{H}(\Omega, \Omega')$?

(34) Let $\mathcal{F} \subset \mathcal{C}(\Omega)$ be normal in a domain Ω. Suppose that $\mathcal{G} \subset \mathcal{H}(\Omega)$ enjoys the following properties:
(i) For each $g \in \mathcal{G}$, $g' \in \mathcal{F}$.
(ii) There is a point $a \in \Omega$ such that $\{g(a) \mid g \in \mathcal{G}\}$ is a bounded set in $\mathbb{C}$.
Prove that $\mathcal{G}$ is normal in Ω.

(35) Read pp.219-227 (skip §5.2) in Ahlfors and do problems: 1-4 on p.227.

(36) Let $A \subset \Omega$ with Ω a domain in $\mathbb{C}$ and A having an accumulation point in Ω. Suppose $\mathcal{F} \subset \mathcal{H}(\Omega)$ is normal in Ω. Let $(f_n)_1^\infty$ be a sequence of in $\mathcal{F}$. Suppose that $(f_n)_1^\infty$ is pointwise convergent on A. Prove that $(f_n)_1^\infty$ converges normally in Ω.

(37) (a) Prove that $f(z) := \displaystyle\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}$ is holomorphic in $\Omega := \mathbb{C} \setminus i\mathbb{Z}_*$, $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$.
(Confirm that the series converges normally in Ω.)
(b) Show that for each $r \in \mathbb{R}_+ \setminus \mathbb{N}$,

$$\int_{C(0;r)} f(z)\, dz = 0\,.$$

(c) Show that for all $m \in \mathbb{N}$ and each $r \in \mathbb{R}_+ \setminus \mathbb{N}$,

$$\int_{C(m;r)} f(z)\, dz = \pi \sum_{|m-r|<n<m+r} \frac{1}{n}\,.$$

(d) Prove that $F(z) := \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{ArcTan}\left(\frac{z}{n}\right)$ is a holomorphic antiderivative for f in the domain $\mathbb{C} \setminus \{iy \mid |y| \geq 1\}$.

(38) Read pp.229-232 in Ahlfors and do problems: 1-2 on p.232.

(39) The first step in our proof of the Riemann Mapping Theorem can be done in a slightly different manner. Instead of using a branch of some square root, we could instead start with a branch of some logarithm. Composing this with an appropriate Möbius transformation would then provide a map into the unit disk sending the base point a to the origin. Fill in the details for this approach.

(40) Reread pp.89-96 in Ahlfors and do problems: 1-8 on pp.96-97.

(41) Everywhere below $w \mapsto \sigma(w)$ is the branch of square root function that is holomorphic for $w \in \mathbb{C} \setminus [0, +\infty)$ with $\sigma(-1) = i$.
(a) Examine in detail the map $z \mapsto \sigma(z^2 - 1)$ for z in the upper half-plane $i\mathbb{H}$. Explain why this map is conformal. Find the image of $i\mathbb{H}$ as well as the "image" of $\mathbb{R}$.
(b) Provide a similar analysis for the map $z \mapsto \sigma(z^2 + 1)$. For what z is this map even defined? When is it conformal? What is the "image" of $\{iy \mid 0 \leq y \leq 1\}$?
(c) What is the connection between the maps in parts (a) and (b)?

(42) Let $c := a + i\,b$ be a point in the upper half-plane. Let K be the circle in $\hat{\mathbb{C}}$ that passes through 0 and c and is orthogonal to $\mathbb{R}$. Let A be the closed subarc of K that joins 0 to c in $i\mathbb{H} \cup \{0\}$. Find a conformal map from $i\mathbb{H} \setminus A$ onto $i\mathbb{H}$. (Suggestion: Start with the Möbius transformation $w = T_c(z) := bz/(a^2 + b^2 - az)$.)

(43) Coming soon!

(44) Let $f \in \mathcal{H}(\mathbb{D})$. Suppose there is a non-empty arc $A \subset \mathbb{T}$ (either open or closed or neither) with the property that f has a continuous extension to $\mathbb{D} \cup A$ and for each $\zeta \in A$, $f(\zeta) = 0$. Prove that $f = 0$. (Hint: Rotate!)

(45) Let $-1 < a < b < 1$. Find a conformal map from $\mathbb{D} \setminus ((-1, a] \cup [b, 1))$ onto $\mathbb{D}$. If possible, do this so that f preserves symmetry as much as possible. What point is mapped to the origin? (Suggestion: First invert. Then 'squash'.)

(46) Let $\Omega \subsetneq \mathbb{C}$ be simply connected. Suppose that Ω is symmetric with respect to some $K \in \hat{\mathcal{C}}$. Prove that any Riemann map (i.e., conformal map) $f : \mathbb{D} \to \Omega$ with $f(0) \in K$ and "$f'(0)$ tangent to K at $f(0)$" is also symmetric with respect to K (in the sense that for all $\zeta \in \mathbb{D}$, $f(\bar{\zeta}) = \rho_K(f(\zeta))$. (Suggestion: Start with the case $K = \hat{\mathbb{R}}$.)

(47) Let K and $\tilde{K}$ be circles in $\hat{\mathbb{C}}$ with associated reflections ρ and $\tilde{\rho}$. Let $f \in \mathcal{H}(\Omega)$. Put $\Omega' := \rho(\Omega)$ and define $g : \Omega' \to \hat{\mathbb{C}}$ by $g(z) := \tilde{\rho} \circ f \circ \rho(z)$. Verify that g is meromorphic in Ω' and describe precisely the set of poles of g. Also, give the multiplicity for g at each of its poles. What if $f \in \mathcal{M}(\Omega)$?

(48) What "happens" if we do *Schwarz Reflection* starting with a meromorphic map (instead of a holomorphic one)?

(49) Let f be an entire function.

(a) Suppose there is an interval $I \subset \mathbb{R}$ such that for all $x \in I$, $f(x) \in \mathbb{R}$. Prove that $f(\mathbb{R}) \subset \mathbb{R}$.

(b) Suppose there is also an interval $J \subset i\mathbb{R}$ such that for all $iy \in J$, $f(iy) \in \mathbb{R}$. Prove that for all $z \in \mathbb{C}$, $f(z) = f(-z)$.

(50) Let $\omega > 0$ and $\Sigma := \{z \in \mathbb{C} \mid 0 < \mathfrak{Im}(z) < \omega\}$. The Riemann Mapping Theorem provides a conformal map $f : \Sigma \to i\mathbb{H}$. In fact, we can choose such a map that will have a homeomorphic extension $f : \overline{\Sigma} \to \overline{i\mathbb{H}} \setminus \{0\}$ with

$$f(\mathbb{R}) = (0, +\infty) \quad \text{and} \quad f(\mathbb{R} + i\omega) = (-\infty, 0)\,.$$

Use the Schwarz Reflection Principle to extend f to an entire map F. What is the range $F(\mathbb{C})$? What is the image of any vertical line? Show that F is periodic and determine a period. Can you identify the map F? What if ω is something "good to eat"?

(51) Let $\tau > 0$ and $\Sigma := \{z \in \mathbb{C} \mid 0 < \mathfrak{Re}(z) < \tau, \mathfrak{Im}(z) < 0\}$. The Riemann Mapping Theorem provides a conformal map $f : \Sigma \to i\mathbb{H}$. In fact, we can choose such a map that will have a homeomorphic extension $f : \overline{\Sigma} \to \overline{i\mathbb{H}}$ with (see Figure 1)

$$f(\{iy \mid y \le 0\}) = [1, +\infty)\ ,\ f(\{\tau + iy \mid y \le 0\}) = (-\infty, -1]\ ,\ f([0, \tau]) = [-1, 1]\,.$$

Use the Schwarz Reflection Principle to extend f to an entire map F. What is the range $F(\mathbb{C})$? Show that F is periodic and determine a period. Can you identify the map F? What if τ is something "good to eat"?

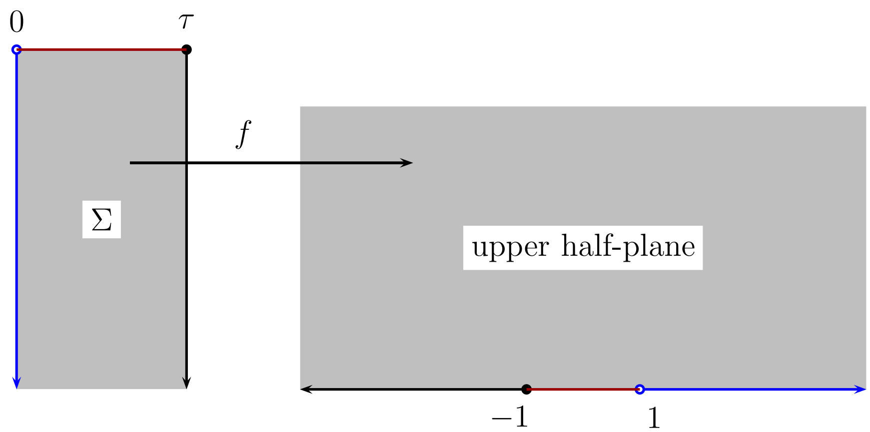

FIGURE 1. Mapping an semi-infinite strip

(52) Let $a > 0$ and $b > 0$. Put $\Omega = (0, a) \times (0, ib) = \{x + iy : 0 < x < a, 0 < y < b\}$ and $Z = \{ma + inb : m, n \in \mathbb{Z}\}$. Suppose $f \in \mathcal{H}(\Omega)$ and maps Ω conformally onto a half-plane. Construct a function $F \in \mathcal{H}(\mathbb{C} \setminus Z))$ satisfying $F|_\Omega \equiv f$.

(53) Prove that two rectangles are conformally equivalent via a vertex preserving homeomorphism if and only if the ratio of their dimensions is exactly the same.

(54) Determine when two annuli are conformally equivalent. (You may assume that any conformal map between two annuli extends to a homeomorphism between their closures.)

(55) Let Ω be the open square with vertices at ± 1 and $\pm i$. Let $\Omega \xrightarrow{f} \mathbb{D}$ be conformal with $f(0) = 0$ and $\theta := \text{Arg}(f'(0))$. Determine the images of $(-1, 1)$, $(-i, i)$ and X under f, where X is the union of the two open line intervals (line segments) that join the opposite midpoints of the edges of $\partial\Omega$.

(56) Find a conformal map from the upper half-plane onto the interior of an equilateral triangle. Explicitly determine the mapping constants in terms of the side length of the given triangle.

(57) Let $a > 0$, $b > 0$. Find a conformal map from the upper half-plane onto $\{u + iv : u > 0, v > 0, (u/a) + (v/b) < 1\}$ as illustrated in Figure 2.

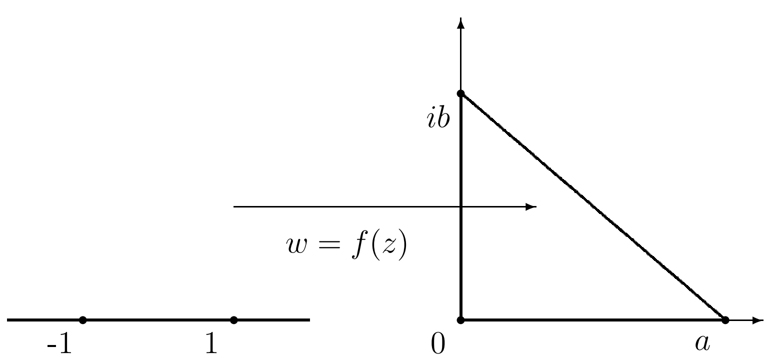

FIGURE 2. Mapping to a right triangle

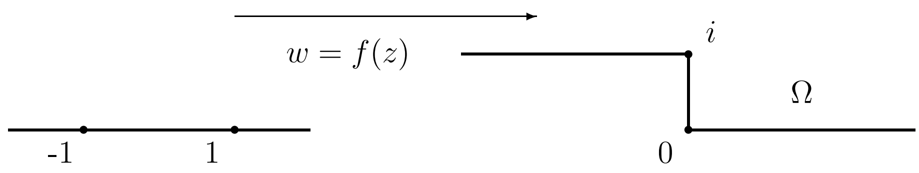

FIGURE 3. Mapping to a 'broken' half-plane

(58) Let $i\mathbb{H} \xrightarrow{f} \Omega$ be the conformal mapping from the upper half-plane onto the region Ω as pictured in Figure 3 with f mappping $-1, 1, \infty$ to $i, 0, \infty$ respectively. Show that
$$f(z) = \frac{1}{\pi}\left[\sqrt{z^2-1} + \operatorname{Log}\left(z + \sqrt{z^2-1}\right)\right].$$

(59) Show that $z \mapsto \displaystyle\int_0^z \frac{d\zeta}{[\zeta(1-\zeta^2)]^{1/2}}$ maps the upper half-plane onto an open square of edge length one.

(60) Show that $z \mapsto \displaystyle\int_0^z \frac{d\zeta}{(1-\zeta^4)^{1/2}}$ maps the unit disk onto an open square.

(61) Show that $z \mapsto \displaystyle\int_0^z \frac{d\zeta}{(1-\zeta^n)^{2/n}}$ maps the unit disk onto an open regular n-gon.

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